**Instructor:** Samuel Talkington **Date Of Lesson:** 09/01/202X

**Content Area and Summary:** This content is part of the power engineering sub-field of electrical engineering. This lesson aims to empower students to apply techniques from linear algebra and vector calculus to develop *practical* approximations of the power flow equations that can be useful when modeling the grid on a computer.

**Course Title and Grade Level:** ECE 63XX, Computational and Statistical Power Flow Analysis; intro graduate

**Lesson Learning Goals:**

The learning goals of this lesson are:

1. Motivate the use of representing complex power as a function of voltage via the power flow equations, in contrast with the linear current-voltage formulations students may be familiar with from undergraduate courses.
2. Introduce students to the matrix-based formulation of the bus injection model of the power flow equations.
3. Introduce students to the concept of the power flow solution manifold.
4. Provide an authentic application of the above two concept items. We will discuss how to apply the above two content items with vector calculus techniques to linearize the power flow equations, producing two practical linear models that can allow for efficient power flow computations.

**Assessments:**

This lesson is intended to be given early in the term; therefore, it is primarily only relevant to **formative** assessments.

- (1) **In-class puzzle:** In the later half of the lesson, students pair off into groups and discuss how to rewrite the complex-valued power flow equations into polar form.
- (2) **Reproducible programming notebook:** Students are asked to implement the results of the discussion in class in a take-home style assessment, simulating the power flow equations for a general network. Learning technology will be leveraged in this assessment, as students will incorporate this puzzle as a part of a **reproducible programming notebook portfolio** that leverages the Binder reproducible code-sharing platform, which is based off of the Pluto notebook framework for the Julia programming language.
- (3) **Discussion-based assessment:** An additional in-class assessment is enabled by observation of the students response to discussion questions, which will contribute to a participation grade.
- (4) **Project-based assessment:** It is expected that this lesson will be integral to the majority of the course projects developed by students of this course. If this content is relevant to their course projects, students also have an opportunity to demonstrate meeting the above Learning Objectives through said course project(s).

## Instructional Strategies and Proactive Management:

- (1) **Early access to content:** The material in this lesson will be made available early to students in an accessible, complete set of lecture notes on Canvas or a similar platform. Expectations for pre-reading will be communicated to students from the beginning of the semester.
- (2) **Use of learning technology:** A key part of this lesson, and the course in general, will be testing the results on real power network data, such as the Texasm A&M University ARPA-e PERFORM synthetic grids, which emulate real power grids in North America. In particular, the Los Alamos National Laboratory PowerModels simulation framework in the Julia programming language will be used. Students will make their implementations of the material available in a reproducible notebook platform such as Binder.
- (3) **Active Student Engagement:** There are multiple opportunities for active students engagement throughout this lesson, in the form of discussion pauses and a pair-and-share exercise.
- (4) **Universal Design for Learning (UDL):** The lesson will incorporate UDL principles. In particular, the lesson will utilize the "Plus-One" framework for inclusive teaching in the following way:
  - A frequent "pinch point" in this lesson is the understanding that the power flow equations are non-linear, in contrast with the linear circuit laws that often have greater emphasis in undergraduate courses which students may have been exposed to.
  - To address this challenge, the instructor will release multiple resources to visualize this non-linearity outside of the traditional mathematical explanation. There will be a 3D-visualization with a simple power grid that shows the non-linearity of the power flow manifold, allowing students to interact with the parameters of a real power grid and get a hands-on look at this property.
  - There will be multiple opportunities to demonstrate understanding of the nonlinearity, which may be application-focused through code, or visual-based through course project presentations.
  - Multiple ways for students to communicate with the instructor and with one another about this concept will be provided around the time this content is introduced.
  - Multiple ways of engaging with the challenges of the non-linearity of the power flow equations will be presented to students, including through society-oriented perspectives (e.g., what can happen to electricity prices if we assume that the equations are linear? Is this fair to all communities involved?)

## Interactive Questions:

### Convergent Questions:

- (1) What is special about Ohm's law? (Bloom–**Remember**: Linear in the voltages)
- (2) What is special about the complex power equation? (Bloom–**Understand**: Non-linear in the voltages)
- (3) How could we generalize to any $n$-node circuit, like a power grid? (Bloom–**Apply**: Admittance matrix)
- (4) How can we make the non-linear power flow equations more like Ohm's law? (Bloom–**Analyze**: Taylor series linearization)
- (5) What is special about the practical linear power flow model? (Bloom–**Evaluate**: Defined by Graph Laplacian matrices)
- (6) How can this make a power flow program more practical? (Bloom–**Create**: Replace nonconvex constraints with linear.)

### Divergent Questions:

- (1) How does the idea of the nodal admittance matrix relate to other areas of math? (Graph Laplacian matrix)
- (2) What if we want to use rectangular voltage coordinates instead of polar?
- (3) How does the linear power flow model relate with the DC OPF model that is popular in industry?
- (4) What are the potential shortcomings of the linear power flow model?
- (5) Is it possible that the use of linear power flow models in industrial settings could have adverse societal effects?

## Beginning Of The Lesson:

### Activate prior knowledge:

**Review:** Linear circuits, Ohm's law, complex power, intro circuit material.
**Preview:** Generalize linear circuits to $n$-node circuits.
**Hook:** Can we use this generalization to model the **entire power grid** with math?

#### Minute 0-3

**Pacing:** Slow, asking students to remember circuits concepts from their undergraduate engineering education. Ensuring to cover the fundamentals enough for those who may not have had a power engineering concentration.

**Content**

Recall from your undergraduate physics class that a resistive circuit is governed by Ohm's law, $V = IR$, where $V$ is voltage, $I$ is current, and $R$ is resistance. The big breakthrough you made in Circuits 2 was that any linear circuit, that is, a circuit with only resistors, inductors, and capacitors, can be represented through a complex-valued form of Ohm's law! We can write

$$V = IZ,$$

where $V, I \in \mathbb{C}$ are complex-valued voltage and current phasors, respectively. Similarly, $Z = R + \mathrm{j}X$ is the complex impedance, where $R$ is the resistance and $X$ is the reactance.

**Directions/Activity**

Group discussion about how this could be generalized to a larger circuit and what happens if we invert impedance.

## The Lesson:

#### Minute 3-6

**Transition:** Discuss 1-dimensional circuits before transitioning to $n$-dimensional graph circuits.

**Content**

You may remember that there is also a quantity known as the admittance, which is the reciprocal of impedance, i.e.,

$$Y = \frac{1}{Z} = \frac{Z^*}{|Z|^2} = \frac{R - \mathrm{j}X}{R^2 + X^2} = \underbrace{\frac{R}{R^2 + X^2}}_{:=G} + \mathrm{j}\underbrace{\frac{-X}{R^2 + X^2}}_{:=B} := G + \mathrm{j}B,$$

where we call $G$ the conductance and $B$ the susceptance!

#### Minute 6-10

**Pacing:**

**Transition:**

**Content(Problem)** The complex power flow equations for linear circuits are non-linear in the voltage phasors:

$$S = VI^* = V\left(\frac{V}{Z}\right)^* = \frac{|V|^2}{Z^*}.$$

**Directions/Activity**

Emphasize how to divide complex numbers, frequent pain point.

#### Minute 10-19

**Pacing:** Quick, should be recollection stage still.

**Transition:** Reminder that circuits are time varying, motivating use of complex numbers.

**Content(Problem)** The bus admittance matrix generalizes the concept of admittance $Y$ concept to $n$ dimensions, for a circuit with any size. It is a complex valued matrix $\mathbf{Y} \in \mathbb{C}^{n \times n}$ with entries

$$\mathbf{Y}_{ij} = \begin{cases} \sum_{k \neq i} y_{ik} & i = j \\ -y_{ij} & i \neq j \end{cases} \quad \text{for all} \quad i, j = 1, \dots, n.$$

**Directions/Activity**

## Minute 20-29

**Pacing:** Quicker, getting up to speed.

**Transition:** Reminder that circuits are time varying and may have many nodes.

**Content(Problem)** Similarly to the one-dimensional equations, we can write Ohm's law in vectorized form. First, define complex-valued vectors $\boldsymbol{s}, \boldsymbol{u}, \boldsymbol{f} \in \mathbb{C}^n$, of power injections, voltage phasors, and current phasors, respectively, which have entries of the form

$$\boldsymbol{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Then, Ohm's law can be written for any $n$-node linear circuit as

$$\boldsymbol{f} = \mathbf{Y}\boldsymbol{u},$$

where $\mathbf{Y} \in \mathbb{C}^{n \times n}$ is the *nodal admittance matrix* that we introduced just a moment ago.

**Directions/Activity**

## Minute 30-34

**Pacing:**

**Transition:**

**Content:** Similarly to the $n$-node Ohm's law, we can write the $n$-node complex power injections as a non-linear system of $n$ equations. First, we need to define the operator $\mathsf{diag}(\cdot)$, which forms a *diagonal* matrix out of any vector you supply to the argument:

$$\mathsf{diag}(\boldsymbol{x}) = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_n \end{bmatrix}, \quad \text{for any} \quad \boldsymbol{x} \in \mathbb{C}^n \quad \text{or} \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Now, we can analyze the entire network's complex power injections via a nonlinear system of equations $\boldsymbol{s} : \mathbb{C}^n \to \mathbb{C}^n$, which takes the form:

$$\boldsymbol{s} = \mathsf{diag}(\boldsymbol{u})\overline{\mathbf{Y}\boldsymbol{u}},$$

where $\overline{(\cdot)}$ denotes the elementwise complex conjugate. This matrix-valued expression captures the complex power injections (pause and remark that this is the bus injection model) for <u>any</u> $n$-node power network.

**Engagement/Active Learning:**
Most computer simulation software can't easily represent complex vectors $\boldsymbol{s}$. So, it is often useful to represent the vectorized power flow equations as a function of the <u>polar form</u> of the voltage phasors; that is, a $2n$-dimensional vector containing voltage magnitudes and phase angles: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\theta} \end{bmatrix}$.

<u>Pair-and-share (10m):</u> How can you write the vectorized equations in polar form? After a few minutes, the instructor(s) should check for understanding, particularly from students not in this major. The instructor should encourage use of Euler's formula, which is the main technique required for this coordinate transformation.

After hearing from students, the instructor should elaborate that <u>one way</u> to transform the matrix-based power flow equations to polar form is

$$\boldsymbol{s}(\boldsymbol{v}, \boldsymbol{\theta}) = \mathsf{diag}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right) \overline{\mathbf{Y}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right)}.$$

The instructor should emphasize that this is <u>not the only way</u> to perform this coordinate transformation. Other approaches used by students, such as representing the equations in trigonometric forms, should be emphasized and discussed, which may potentially lead to the rest of the content being covered in another discussion session.

## Minute 35-39

**Pacing:**

**Transition:** The instructor explains that students can use the polar coordinate transformation that they developed during the pair-and-share activity to separate the real and reactive power components of the equations as

$$\begin{bmatrix} \boldsymbol{p}(\boldsymbol{v}, \boldsymbol{\theta}) \\ \boldsymbol{q}(\boldsymbol{v}, \boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} \mathsf{Re}\left(\mathsf{diag}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right) \overline{\mathbf{Y}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right)}\right) \\ \mathsf{Im}\left(\mathsf{diag}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right) \overline{\mathbf{Y}\left(\boldsymbol{v} \circ \exp(\mathrm{j}\boldsymbol{\theta})\right)}\right) \end{bmatrix}.$$

**Content** The instructor then asks students to consider what the set of all solutions to the power flow equations might look like. The instructor should emphasize that this is a very mysterious concept that cannot be easily visualized, but they can use the tools they are developing to construct an abstract power flow solution manifold.
**Intellectual diversity:** The instructor should emphasize that a "manifold" is a mathematical concept that refers to the vector subspace of $\mathbb{C}^n$ that satisfies a nonlinear system of equations. In our case, this is the set of all solutions to the power flow equations $\boldsymbol{s} : \mathbb{C}^n \to \mathbb{C}^n$ for any fixed, or nominal value of voltage phasors $\boldsymbol{u}^\bullet$, which produces the desired power injections $\boldsymbol{s}(\boldsymbol{u}^\bullet)$. The instructor should encourage students to write the equation

$$\mathcal{M}_{\boldsymbol{u}^\bullet} = \{\boldsymbol{u} \in \mathbb{C}^n \ : \ \boldsymbol{s}(\boldsymbol{u}^\bullet) - \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}\}\,.$$

This is practically useful because we can linearize the power flow solution manifold around $\boldsymbol{u}^\bullet$. In particular, we often want to maintain a nominal voltage of $\boldsymbol{u}^\bullet = \boldsymbol{1} + \mathrm{j}\boldsymbol{0}$; this is known as the flat start condition. If we perform a first-order Taylor series linearization around this "flat start" voltage, we obtain

$$\begin{bmatrix} \boldsymbol{p}(\boldsymbol{v},\boldsymbol{\theta}) \\ \boldsymbol{q}(\boldsymbol{v},\boldsymbol{\theta}) \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{p}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) \\ \boldsymbol{q}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) \end{bmatrix} + \begin{bmatrix} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) & \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) \\ \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) & \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}(\boldsymbol{v}^\bullet,\boldsymbol{\theta}^\bullet) \end{bmatrix} \begin{bmatrix} \boldsymbol{v} - \boldsymbol{1} \\ \boldsymbol{\theta} \end{bmatrix},$$

simplifying each of the blocks of the Jacobian matrix yields

$$\begin{bmatrix} \boldsymbol{p}(\boldsymbol{v},\boldsymbol{\theta}) \\ \boldsymbol{q}(\boldsymbol{v},\boldsymbol{\theta}) \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} - \boldsymbol{1} \\ \boldsymbol{\theta} \end{bmatrix},$$

where $\boldsymbol{G}$, $\boldsymbol{B}$ are the real and imaginary components of the nodal admittance matrix $\boldsymbol{Y}$, respectively!

**Directions/Activity** Intellectual diversity: The instructor will ask if any students in the class are returning to school from the Power Engineering industry. If such students are present, the instructor will ask if this looks like the familiar "DC" power flow approximation that is used in industrial settings.

## Closure:

**Content Summary:** To summarize, we introduced a powerful way to understand the non-linear power flow equations through practical linear approximations.
The lesson will conclude with a group discussion of the benefits and challenges of such a model, with a particular focus on the **societal risks** associated with using this model–e.g., how can linearizing the power flow equations lead to unfair outcomes in electricity prices? Or, how can it lead to increased risks of power blackouts and costs to communities, in exchange for operational efficiency? Is there a better way we can do things? In contrast, what are the benefits and improvements gained from this approach?