Computational Power Flow, Sample Exam

ECE 63XX
10/XX/2026

Name: ________________________________

I, ________________________________, commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech Community.

(Revised) **Please read this information:**

- This is a 48-hour take home exam.
- Please do not collaborate during your *first revision* of this exam. You are on your honor.
- Collaboration *may* be permitted for a second revision submission for partial credit.
- Use of generative AI is discouraged due to the nature of the material being creative; reflect on the syllabus policy on use of AI.
- You are responsible for the content of all your answers.
- Please show all your work.
- Please box or circle your final answers.
- This test has 5 problems that total up to 100 points.

Exam wrapper (3 bonus points) (Revised)

Question I. (1 pts)

Reflect on your work in preparation for this course by answering the following questions:

1. Approximately how many hours did you spend studying for this exam? _______
2. Please indicate what percentage of your time was spent on the components of the course:
   (a) Prepared course notes: _______
   (b) Lecture slides and handwritten notes: _______
   (c) Solving and resolving homework: _______
   (d) Researching material on my own: _______

Question II. (1 pts)

Reflect on the topics you believed were your strengths and weaknesses going into this exam. You don't need to use every blank space.

1. Which topic(s) did you feel the most confident about?
   (a) ____
   (b) ____
   (c) ____
2. What topic(s) did you feel the least confident about?
   (a) ____
   (b) ____
   (c) ____

Question III. (1 pts)

Reflect on your interests in this course in preparation for your final project. In your opinion, what was the most interesting part of this course thus far?

____

Item 1: Matrix Methods for Power Flow Analysis (20pts)

Consider the following 3-node electric power network:

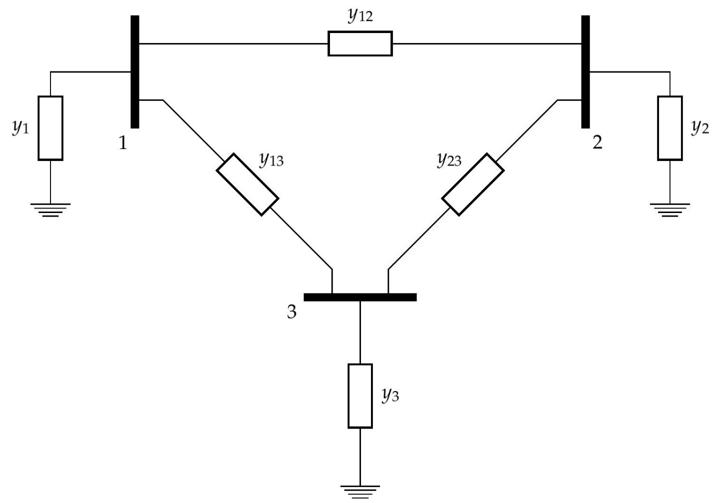

Figure 1: A 3-node electric power network

Question 1a. (10 pts)

Build and write the 3×3 *nodal admittance matrix* Y for this network in terms of the admittance symbols shown for the lines and shunts in Fig. 1.

Solution 1a.

Recall that the nodal admittance matrix is the *graph Laplacian* matrix with the line admittances used to define the graph edge weights. Thus, the matrix Y takes the form

$$Y = \begin{bmatrix} y_1 + \sum_{k \neq 1} y_{1k} & -y_{12} & -y_{13} \\ -y_{21} & y_2 + \sum_{k \neq 2} y_{2k} & -y_{23} \\ -y_{31} & -y_{32} & y_3 + \sum_{k \neq 3} y_{3k} \end{bmatrix} \tag{1}$$

Question 1b. (10 pts)

Derive the power flow equation $s_1 : \mathbb{C}^n \to \mathbb{C}$ for node 1 in the network shown in Fig. 1, which maps the voltages at all nodes in the network to the power injected at node 1. The equation should only depend on the y_{ik}'s and the v_i's.

Solution 1b.

Let $v_i = x_i + \mathrm{j}y_i \in \mathbb{C}$ denote the voltage phasors at each node $i \in \{1, 2, 3\}$ and let $v = [v_i]_i \in \mathbb{C}^3$ be the network state. Let $y_i \in \mathbb{C}^3$ be the i-th row of the admittance matrix Y. Recall that the power flow equations are

$$s(v) = \mathsf{diag}(v)\underline{Y v},$$

where $\underline{(\cdot)}$ denotes the complex conjugate. Hence, we can write this in elementwise matrix-vector form for the 3-node network shown in Fig. 1 as

$$\begin{aligned}
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} &= \begin{bmatrix} v_1 & & \\ \cdot & v_2 & \\ \cdot & \cdot & v_3 \end{bmatrix} \underline{\begin{bmatrix} y_1 + \sum_{k\neq 1} y_{1k} & -y_{12} & -y_{13} \\ -y_{21} & y_2 + \sum_{k\neq 2} y_{2k} & -y_{23} \\ -y_{31} & -y_{32} & y_3 + \sum_{k\neq 3} y_{3k} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} \\
&= \begin{bmatrix} v_1 & & \\ \cdot & v_2 & \\ \cdot & \cdot & v_3 \end{bmatrix} \underline{\begin{bmatrix} \cdots & y_1^\top & \cdots \\ \cdots & y_2^\top & \cdots \\ \cdots & y_3^\top & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ v \\ \vdots \end{bmatrix}}
\end{aligned}$$

Now, we can see that the power flow equations s_i for each node $i \in \{1, 2, 3\}$ are given as

$$s_i = v_i \underline{y_i^\top v} \quad \text{for all} \quad i \in \{1, 2, 3\}.$$

Hence, the equation for node 1 is

$$\begin{aligned}
s_1 &= v_1 \underline{y_1^\top v} \\
&= v_1 \underline{\left(\left(y_1 + \sum_{k\neq 1} y_{1k} \right) v_1 - y_{12} v_2 - y_{13} v_3 \right)} \\
&= v_1 \underline{\left(\left(y_1 + y_{12} + y_{13} \right) v_1 - y_{12} v_2 - y_{13} v_3 \right)}.
\end{aligned}$$

Item 2: Recognize Power Flow Representations (20pts)

Please write in the spaces provided the number that corresponds to the approximation or relaxation of the power flow equations

Question 2a. (5 pts)

Circle your answer: The branch flow model is more appropriate for

1. Single-phase radial distribution networks
2. Single-phase meshed transmission networks

Solution 2a.

The answer is **radial (i.e., tree) distribution networks** because the branch flow model does not enforce consistency in the angle summations around cycles in the network (i.e., the summation of the angles around a cycle must be a multiple of 2π radians for true solutions). **Since transmission network models are typically meshed**, the branch flow model is a relaxation for transmission networks; in contrast, it is *exact* for radial distribution networks.

Question 2b. (5 pts)

Circle your answer: For multi-phase unbalanced radial distribution networks, the branch flow model

1. Is a relaxation of the power flow equations
2. Is an approximation of the power flow equations
3. Is neither a relaxation nor approximation of the power flow equations

Solution 2b.

The answer is that the DistFlow equations are a **relaxation** of the power flow equations for multi-phase *unbalanced* distribution networks, because unbalanced networks have *implicit cycles between phases*; thus, the network cannot be modeled as a single-phase balanced radial network. This is a requirement for the DistFlow equations to be exact; otherwise, it is a relaxation.

Question 2c. (5 pts)

Circle your answer: A node consumes 2 MVA with a leading power factor of 0.8. How much reactive power is the load consuming?

1. 1.2 MVAr
2. -1.2 MVAr
3. 1.6 kVAr
4. -1.6 MVAr

Solution 2c.

The answer is **2, -1.2 MVAR**. The injection is $s = p + jq$ where $|s| = 2MVA$. We can write

$$q = \mathsf{sgn}(q) \cdot \frac{p}{\alpha}\sqrt{1-\alpha^2}$$

where $\alpha \in (0,1)$ is the power factor. Since the power factor is *leading*, $\mathsf{sgn}(q) = -1$ and $p = (0.8) \cdot \left(2 \times 10^6\right) = 1.6 \times 10^6$, so

$$q = -\left(1.6 \times 10^6\right)\frac{\sqrt{1-(0.8)^2}}{0.8} = -1.2 \times 10^6 \text{ VAr} = -1.2 \text{ MVAr}.$$

Question 2d. (5 pts)

For each of the following equations, write TRUE if the equation is either a relaxation or approximation of the power flow equations, and FALSE if it is neither. The equations assume an arbitrary n-node network.

1. $\boldsymbol{p} + \mathrm{j}\boldsymbol{q} = \mathrm{diag}(\boldsymbol{v})\underline{\boldsymbol{Y}\boldsymbol{v}}$, where $\boldsymbol{Y} \in \mathbb{C}^{n \times n}$, $\boldsymbol{p}, \boldsymbol{q}, \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{C}^n$.
2. $\boldsymbol{v} = \boldsymbol{1} + \boldsymbol{R}\boldsymbol{p} + \boldsymbol{X}\boldsymbol{q}$, where $\boldsymbol{R}, \boldsymbol{X} \in \mathbb{S}^n_+$, and $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$
3. $\boldsymbol{p} = \boldsymbol{B}\boldsymbol{\theta}$, where $\boldsymbol{B} \in \mathbb{S}^n_+$, and $\boldsymbol{p}, \boldsymbol{\theta} \in \mathbb{R}^n$
4. $f_i = \sum_{k=1}^n Y_{ik} v_k$
5. $p_i + \mathrm{j}q_i = v_i \sum_{k=1}^n \underline{Y_{ik}}\,\underline{v_k} \quad \forall i \in \mathcal{N}$
6. $p_i = |v_i| \sum_{k=1}^n |v_k| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k)\right)$
7. $p_k = \mathsf{trc}(\boldsymbol{H}_k \boldsymbol{W})$, where $\boldsymbol{H}_k := \frac{1}{2}\left(\boldsymbol{Y}^* \boldsymbol{e}_k \boldsymbol{e}_k^\top + \boldsymbol{e}_k \boldsymbol{e}_k^\top \boldsymbol{Y}\right)$ and $\boldsymbol{W} \succeq 0$.
8. $p_k = \mathsf{trc}(\boldsymbol{H}_k \boldsymbol{W})$, where $\boldsymbol{H}_k := \frac{1}{2}\left(\boldsymbol{Y}^* \boldsymbol{e}_k \boldsymbol{e}_k^\top + \boldsymbol{e}_k \boldsymbol{e}_k^\top \boldsymbol{Y}\right)$ and $\mathsf{rank}(\boldsymbol{W}) = 1$.
9. $$\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \begin{bmatrix} |\boldsymbol{v}| - \boldsymbol{1} \\ \boldsymbol{\theta} \end{bmatrix}$$
10. $p_{ik} = g_{ik}\left(T_{ik} + U_{ik} + |v_i| - |v_k|\right) - b_{ik}\left(\theta_i - \theta_k\right)$, where $U_{ik} \geq \left(|v_i| - |v_k|\right)^2$ and $T_{ik} \geq |\theta_i - \theta_k|^2$ for all $(i,k) \in \mathcal{E}$

Solution 2d.

1. **Relaxations:** 7
2. **Approximations:** 2,3,9,10
3. **Neither:** 1,4,5,6,8

Item 3: Predict Grid Conditions

Question 3a. (20 pts)

A solar photovoltaic device applies a DC voltage v in parallel across two loads with parameters $R_1 := 1\Omega$ and $R_2 := 2\Omega$. Assume that you obtain random measurements from the network of the form

$$f_1 \sim \mathcal{N}(25, 1), \quad f_2 \sim \mathcal{N}(11, 1/4),$$

where f_1 is the current through R_1 and f_2 is the current through R_2. Derive the minimum mean squared error (MMSE) unbiased estimator for v.

Solution 3a.

Let $\boldsymbol{x} = [v]$ be the DC voltage as the state. Applying the power flow equations, the measurement vector is

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} h_1(\boldsymbol{x}) \\ h_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \frac{v}{1\Omega} \\ \frac{v}{2\Omega} \end{bmatrix}.$$

The linearization of the measurement operator is

$$\boldsymbol{H} := \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

The Gram matrix is

$$\boldsymbol{G} = \boldsymbol{H}^\top \boldsymbol{R}^{-1} \boldsymbol{H} = \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = 2$$

The MMSE estimator is

$$\hat{\boldsymbol{x}} = \boldsymbol{G}^{-1} \boldsymbol{H}^\top \boldsymbol{R}^{-1} (\boldsymbol{z} - \boldsymbol{h}(\boldsymbol{x}_0)) = \frac{1}{2} \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 4 \end{bmatrix} \begin{bmatrix} 25 \\ 11 \end{bmatrix} = \frac{25 + 22}{2} = 23.5 \text{ V}.$$

Item 4: Economic Power Dispatch

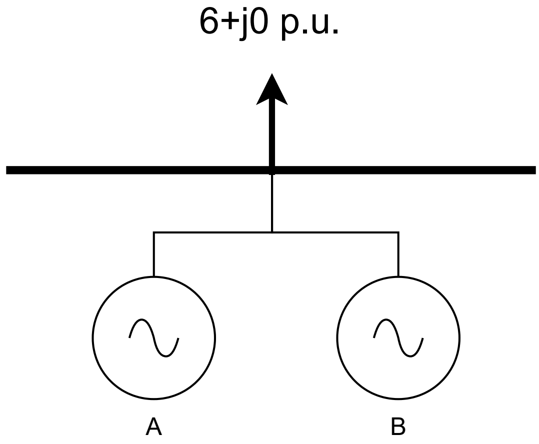

Figure 2: A 1-node network serving load $s = 6 + \mathrm{j}0$ p.u.

Consider a single node network shown in Fig. 2. The cost curves for each generator are given as

$$c_a = \frac{1}{6}p_{g,a}^3 \qquad 0 \le p_{g,a} \le 5 \tag{2a}$$

$$c_b = \frac{1}{6}p_{g,b}^3 + 3p_{g,b} \qquad 0 \le p_{g,b} \le 5. \tag{2b}$$

Question 4a. (10 pts)

Determine the optimal generations $p_a^{g\star}$ and $p_b^{g\star}$ that minimize the total operating cost.

Solution 4a.

We have that the total operating cost (objective function) of the program $C : \mathbb{R}^2 \to \mathbb{R}$ is given as

$$C(\boldsymbol{p}) = \frac{1}{6}p_{g,a}^3 + \frac{1}{6}p_{g,b}^3 + 3p_{g,b}.$$

The incremental cost vector is

$$\boldsymbol{c} = \nabla_{\boldsymbol{p}} C = \begin{bmatrix} \frac{1}{3}p_{g,a}^2 \\ \frac{1}{3}p_{g,b}^2 + 3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$$

Moreover, the power balance constraint is given as

$$p_{g,a} + p_{g,b} = 6 \implies p_{g,b} = 6 - p_{g,a}.$$

So, combining the above with the incremental cost vector $\boldsymbol{c}$ yields

$$\frac{1}{3}p_{g,a}^2 = \frac{1}{3}p_{g,b}^2 + 3 = \frac{1}{3}\left(6 - p_{g,a}\right)^2 + 3.$$

Rearranging, we have

$$\frac{1}{3}p_{g,a}^2 = \frac{1}{3}\left(6 - p_{g,a}\right)^2 + 3 \implies p_{g,a}^2 = \left(6 - p_{g,a}\right)^2 + 9 \implies p_{g,a}^2 = 39 - 12p_{g,a} + p_{g,a}^2;$$

thus, we have

$$-39 = -12p_{g,a} \implies p_{g,a} = \frac{39}{12} \implies p_{g,b} = 6 - \frac{39}{12} = \frac{11}{4}.$$

Question 4b. (10 pts)

Determine the locational marginal price (LMP) of power served optimally at the node.

Solution 4b.

The LMP is given as

$$\lambda = \frac{1}{3}p_{g,a}^2 = \frac{1}{3}\left(\frac{39}{12}\right)^2 = 3.52 = \$3,520/\mathsf{MWh}$$

Question 4c. (10 pts)

Determine the total operating cost in $\$/hr$.

Solution 4c.

The total operating cost is

$$C(\boldsymbol{p}^\star) = \frac{1}{6}p_{g,a}^{\star 3} + \frac{1}{6}p_{g,b}^{\star 3} + 3p_{g,b}^\star = 17.4375 = \$17,437.50/\mathsf{MWhr}$$

Item 5: Control grid operating conditions

Question 5a. (30 pts)

Consider a single-phase radial distribution network with n nodes. Assume that:

1. There are solar panels installed at every node with a maximum power output of Δ.
2. The LinDistFlow approximation accurately models the grid.
3. Reactive power injections are 0 throughout the grid ($\boldsymbol{q} = \boldsymbol{0}$).

Derive an upper bound for maximum expected random voltage magnitude perturbation from $\mathbf{1}$ anywhere in the grid, that depends only on the rows of the resistance matrix $\{\boldsymbol{r}_i\}_i$, the number of nodes n, and the maximum solar panel output Δ.

Solution 5a.

Under the problem assumptions, the random voltage magnitude perturbations around $\mathbf{1}$ are given as

$$\boldsymbol{v} - \mathbf{1} = \boldsymbol{R}\boldsymbol{p},$$

where $\boldsymbol{p}$ is a random vector such that $\|\boldsymbol{p}\|_\infty \leq \Delta$ almost surely. Consequently, it follows that $\boldsymbol{p}$ is a sub-Gaussian random vector with parameter at most $\Delta/2$; hence, the maximum random voltage perturbations can be upper bounded as

$$\begin{aligned}
\mathsf{E}\left[\max_{i=1,\ldots,n} |v_i - 1|\right] &\overset{(1)}{=} \mathsf{E}\left[\|\boldsymbol{v} - \mathbf{1}\|_\infty\right] \\
&\overset{(2)}{=} \mathsf{E}\left[\|\boldsymbol{R}\boldsymbol{p}\|_\infty\right] \\
&\overset{(3)}{\leq} \|\boldsymbol{R}\|_\infty \, \mathsf{E}\left[\|\boldsymbol{p}\|_\infty\right] \\
&\overset{(4)}{\lesssim} \left(\max_{i=1,\ldots,n} \|\boldsymbol{r}_i\|_2\right) \cdot \left(\Delta\sqrt{\log 2n}\right)
\end{aligned}$$

where step (1) is by definition of the ℓ_∞ norm, step (2) is by definition of the LinDistFlow approximation of the power flow equations, step (3) is by the submultiplicative property of any matrix p-norm, and step (4) is by defintion of the matrix ∞ norm and the concentration of the maximum of sub-Gaussian random variables.

ALTERNATIVES:

1. Operator norm-based upper bound is also acceptable.