ECE 4320, Spring 2025
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Homework 5

Due: April 22

4 problems, 100 points possible

1 State Estimation

This problem tasks you with formulating (but not solving) the equations associated with a state estimation problem for the two-bus system shown below in Fig. 1.

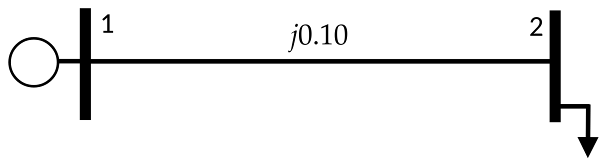

Figure 1: Two bus system

This system has a generator at bus 1 and a load at bus 2. The line is modeled with a series impedance of $j0.10$ per unit. Bus 1 sets the angle reference so $\theta_1 = 0°$. (Note: the phase angle reference is not a measurement, so θ_1 is modeled as an exact value in the formulation below.) For this system, the state vector $\boldsymbol{x}$ contains the voltage angle at bus 2 (θ_2), the voltage magnitude at bus 1 (V_1), and the voltage magnitude at bus 2 (V_2):

$$\boldsymbol{x} = \begin{bmatrix} \theta_2 \\ V_1 \\ V_2 \end{bmatrix}$$

Table 1: One sample of measurements of the system

Measurement	Measured Value (per unit)	Standard Deviation (per unit)
Voltage magnitude at bus 1, V_1	$V_1 = 1.001$	$\sigma_1 = 0.01$
Voltage magnitude at bus 2, V_2	$V_2 = 0.969$	$\sigma_2 = 0.01$
Active power flow on the line, P_{12}	$P_{12} = 1.033$	$\sigma_3 = 0.03$
Reactive power flow on the line, Q_{12}	$Q_{12} = 0.464$	$\sigma_4 = 0.03$

Problem 1 (25pts)

Write the (weighted) least-squares optimization formulation for the state estimation problem that uses the four specified measurements from the table to compute an estimate for the system state $\boldsymbol{x}$. **Note: You do not need to solve this formulation.**

1. Write down the measurement vector $\boldsymbol{z}$ as described in Table 1 and construct the measurement functions $\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} h_1(\boldsymbol{x}) & \ldots & h_4(\boldsymbol{x}) \end{bmatrix}^\top$. [15pts]

2. Write the objective function $C(\boldsymbol{x})$ for the least squares problem. [10pts]

SOLUTION:

First consolidate the measurements into a vector $\boldsymbol{z}$:

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1.001 \\ 0.969 \\ 1.033 \\ 0.464 \end{bmatrix}.$$

We then write the measurement function $\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} h_1(\boldsymbol{x}) & \dots & h_4(\boldsymbol{x}) \end{bmatrix}^\top$. The first two measurements $h_1(\boldsymbol{x})$ are the same as the states V_1 and V_2 themselves. The second two measurements are the active and reactive power flows across the line. We can derive these using Ohm's law, or equivalently via the power flow equations (look back at our notes on the power flow equations!). Using Ohm's law the current through the line is $I_{12} = \frac{(V_1\angle 0^\circ - V_2\angle\theta_2^\circ)}{j0.10} = -j10\,(V_1\angle 0^\circ - V_2\angle\theta_2)$. Thus, the active power flow P_{12} is

$$\begin{aligned} P_{12} &= \mathsf{Re}\left(V_1\angle 0^\circ \cdot I_{12}^*\right) \\ &= \mathsf{Re}\left(j10 \cdot V_1^2\angle 0^\circ - j10 \cdot V_1 V_2 \cos(\theta_2) - 10 \cdot V_1 V_2 \sin(\theta_2)\right) \\ &= -10 \cdot V_1 V_2 \sin(\theta_2). \end{aligned}$$

The reactive power flow Q_{12} is:

$$\begin{aligned} Q_{12} &= \mathsf{Im}\left(V_1\angle 0^\circ \cdot I_{12}^*\right) \\ &= \mathsf{Im}\left(j10 \cdot V_1^2\angle 0^\circ - j10 \cdot V_1 V_2 \cos(\theta_2) - 10 \cdot V_1 V_2 \sin(\theta_2)\right) \\ &= \mathsf{Im}\left(j10 \cdot V_1^2 - j10 \cdot V_1 V_2 \cos(\theta_2) - 10 \cdot V_1 V_2 \sin(\theta_2)\right) \\ &= 10 \cdot V_1^2 - 10 \cdot V_1 V_2 \cos(\theta_2) \\ &= 10 \cdot V_1 \left(V_1 - V_2 \cos(\theta_2)\right). \end{aligned}$$

Now, we can write the WLS problem as

$$\min_{\boldsymbol{x}} \underbrace{\sum_{i=1}^{4} \frac{\left(h_i(\boldsymbol{x}) - z_i\right)^2}{\sigma_i^2}}_{=C(\boldsymbol{x})},$$

where the function $C(\boldsymbol{x})$ is given as

$$\begin{aligned} C(\boldsymbol{x}) &= \sum_{i=1}^{4} \frac{\left(h_i(\boldsymbol{x}) - z_i\right)^2}{\sigma_i^2} \\ &= \frac{1}{\sigma_1^2}\left(h_1(\boldsymbol{x}) - z_1\right)^2 + \dots + \frac{1}{\sigma_4^2}\left(h_4(\boldsymbol{x}) - z_4\right)^2 \\ &= \frac{1}{\sigma_1^2}\left(V_1 - z_1\right)^2 + \frac{1}{\sigma_2^2}\left(V_2 - z_2\right)^2 + \frac{1}{\sigma_3^2}\left(P_{12} - z_3\right)^2 + \frac{1}{\sigma_4^2}\left(Q_{12} - z_4\right)^2 \\ &= \frac{1}{\sigma_1^2}\left(V_1 - z_1\right)^2 + \frac{1}{\sigma_2^2}\left(V_2 - z_2\right)^2 + \frac{1}{\sigma_3^2}\left(-10 \cdot V_1 V_2 \sin(\theta_2) - z_3\right)^2 + \frac{1}{\sigma_4^2}\left(10 \cdot V_1^2 - 10 \cdot V_1 V_2 \cos(\theta_2) - z_4\right)^2. \end{aligned}$$

2 Matrix Methods for Three-Phase Fault Analysis

Note: This problem had some typos in the admittance matrix. We first write the problem and solution as written and then the problem that I intended to have you solve and the associated solution. Sorry for the confusion on this!

Consider the admittance matrix shown below, with values given in per unit.

$$\boldsymbol{Y} = \begin{bmatrix} 0+j22 & 0-j8 & 0-j4 & 0-j10 \\ 0-j8 & 0+j8 & 0+j0 & 0+j0 \\ 0-j4 & 0+j0 & 0+j9.5 & 0-j5.5 \\ 0-j10 & 0+j0 & 0-j5.5 & 1-j12.5 \end{bmatrix}$$

Problem 2 (25pts)

From the admittance matrix $\boldsymbol{Y}$, complete the following tasks:

1. Draw the one-line diagram for the power system that has the admittance matrix $\boldsymbol{Y}$ shown above. On your one-line diagram, label all **impedances** in per-unit representation. [6pts]
2. Compute the impedance matrix $\boldsymbol{Z} = \boldsymbol{Y}^{-1}$ using MATLAB, or your computation method of choice. [3pts]
3. Suppose that a *bolted three-phase fault* occurs at bus 3. Neglect all prefault current and assume a flat prefault voltage of $1\angle 0^\circ$ per unit. Compute the following using the impedance matrix method:
   (a) The current into the fault in per unit. [8pts]
   (b) The voltages during the fault at buses 1, 2, and 4. [8pts]
4. **[BONUS: +3pts]** Explain how to make it impossible to compute $\boldsymbol{Z}$ (i.e., make $\boldsymbol{Y}$ singular) by changing exactly 1 entry of $\boldsymbol{Y}$. Explain what this means physically in 1 or 2 sentences.

SOLUTION:

1. Any one-line diagram that correctly shows the connectivity between all buses and labels the **impedances** on the lines **and the shunt** on bus 4 is okay.

2. We have that

$$\boldsymbol{Z} = \begin{bmatrix} 0.00127 - j0.0455 & 0.00127 - j0.0455 & 0.00127 + j0.00148 & 0.00127 + j0.0357 \\ 0.00127 - j0.0455 & 0.00127 - j0.1705 & 0.00127 + j0.00148 & 0.00127 + j0.0357 \\ 0.00127 + j0.00148 & 0.00127 + j0.00148 & 0.00127 - j0.0840 & 0.00127 + j0.0357 \\ 0.00127 + j0.0357 & 0.00127 + j0.0357 & 0.00127 + j0.0357 & 0.00127 + j0.0357 \end{bmatrix},$$

where we rounded to 3 significant figures.

3. **BONUS:** Note that the entry Y_{44} is not the negative of the sum of the off-diagonal elements:

$$-\sum_{j \neq 4} \boldsymbol{Y}_{4j} = -(0 - j10 + 0 - j5.5) = 0 + j15.5 \neq \boldsymbol{Y}_{44} = 1 - j12.5;$$

this means that there must be a *shunt impedance to ground*. To compute this, recall that

$$\boldsymbol{Y}_{ij} = -y_{ij} \quad \forall i, j = 1, \ldots, N, \quad i \neq j.$$

And recall that the entry $\boldsymbol{Y}_{4,4}$ is given as

$$\boldsymbol{Y}_{4,4} = Y_{\mathsf{sh},4} + \sum_{j \neq 4} y_{4j} = Y_{\mathsf{sh},4} - \sum_{j \neq 4} \boldsymbol{Y}_{4,j}$$

Thus, we can compute the shunt admittance as:

$$\boxed{Y_{\mathsf{sh},4} = \boldsymbol{Y}_{4,4} + \sum_{j \neq 4} \boldsymbol{Y}_{4j} = (1 - j12.5) + (0 - j10) + (0 - j5.5) = 1 - j28 \quad \text{p.u.}}$$

If we **remove the shunt impedance** and set

$$\boxed{\boldsymbol{Y}_{4,4}^{\mathsf{new}} = \boldsymbol{Y}_{4,4} - Y_{\mathsf{sh},4} = 0 + j15 \quad \text{p.u.},}$$

there is **no longer a ground in the circuit**, which consequently makes the admittance matrix singular; namely,

$$\mathsf{null}(\boldsymbol{Y}) = \mathsf{span}(\mathbf{1}),$$

where **1** is a vector of all ones.

Below are the fault solutions, applying the equations from lecture.

1. The current into the fault in per unit is given as:

$$I_f = \frac{1\angle 0^\circ}{Z_{3,3}} = \frac{1\angle 0^\circ}{0.00127 - j0.0840} = \boxed{0.1805 + j11.9035 = 11.905\angle 89.131^\circ}$$

2. Voltages during the fault are given as

$$V_1 = \left(1 - \frac{0.00127 + j0.00148}{0.00127 - j0.0840}\right) = 1.0174 - j0.0153 = 1.018\angle - 0.869^\circ \tag{1}$$

$$V_2 = \left(1 - \frac{0.00127 + j0.00148}{0.00127 - j0.0840}\right) = 1.0174 - j0.01543 = 1.018\angle - 0.869^\circ \tag{2}$$

$$V_4 = \left(1 - \frac{0.00127 + j0.0357}{0.00127 - j0.0840}\right) = 1.424 - j0.0216 = 1.425\angle - 0.869^\circ. \tag{3}$$

The problem I had intended for you to solve was to compute the fault currents for the system with the following admittance matrix:

$$\boldsymbol{Y} = \begin{bmatrix} 0 - j22 & 0 + j8 & 0 + j4 & 0 + j10 \\ 0 + j8 & 0 - j8 & 0 + j0 & 0 + j0 \\ 0 + j4 & 0 + j0 & 0 - j9.5 & 0 + j5.5 \\ 0 + j10 & 0 + j0 & 0 + j5.5 & 1 - j12.5 \end{bmatrix}$$

This admittance matrix corresponds to a system with lines between the following buses:

- Bus 1 and bus 2 with impedance $Z_{12} = 1/(-j8) = j0.125$.
- Bus 1 and bus 3 with impedance $Z_{13} = 1/(-j4) = j0.25$.
- Bus 1 and bus 4 with impedance $Z_{14} = 1/(-j10) = j0.10$.
- Bus 3 and bus 4 with impedance $Z_{34} = 1/(-j4) = j0.1818$.

There is a shunt impedance at bus 4 (observe that the fourth column does not sum to zero). The admittance connected to bus 4 is $(1 - j12) - (-j10 - j5.5) = 1 + j3.5$. The associated impedance is thus $Z_{sh,4} = 1/(1 + j16.5) = 0.0755 - j0.2642$.

Taking the inverse of this matrix yields the impedance matrix $\mathbf{Z}$:

$$\mathbf{Z} = \begin{bmatrix} 0.1 - 0.2188j & 0.1 - 0.2188j & 0.1 - 0.2658j & 0.1 - 0.3j \\ 0.1 - 0.2188j & 0.1 - 0.0938j & 0.1 - 0.2658j & 0.1 - 0.3j \\ 0.1 - 0.2658j & 0.1 - 0.2658j & 0.1 - 0.1803j & 0.1 - 0.3j \\ 0.1 - 0.3j & 0.1 - 0.3j & 0.1 - 0.3j & 0.1 - 0.3j \end{bmatrix}$$

The fault current for a three-phase fault at bus 3 is $I_f = 1\angle 0^\circ / \mathbf{Z}_{33} = \frac{1}{0.1 - j0.1803} = 2.3517 + j4.2410 = 4.8494\angle 61.0^\circ$ per unit.

The voltages at the remaining buses i during the fault at bus k are given by:

$$V_i = 1\angle 0^\circ - \frac{\mathbf{Z}_{ik}}{\mathbf{Z}_{kk}}(1\angle 0^\circ)$$

We therefore have:

$$V_1 = 1\angle 0^\circ - \frac{0.1000 - j0.2658}{0.1000 - j0.1803} = -0.3625 + j0.2010 = 0.4145\angle 151^\circ$$
$$V_2 = 1\angle 0^\circ - \frac{0.1000 - j0.2658}{0.1000 - j0.1803} = -0.3625 + j0.2010 = 0.4145\angle 151^\circ$$
$$V_4 = 1\angle 0^\circ - \frac{0.1000 - j0.3000}{0.1000 - j0.1803} = -0.5075 + j0.2814 = 0.5803\angle 151^\circ$$

3 Analysis of Unbalanced Faults

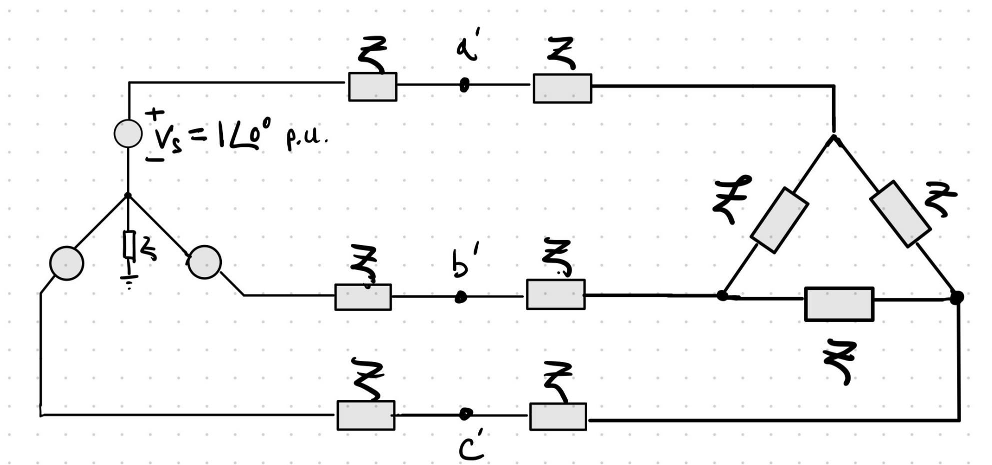

Figure 2: The Problem 3 system

Consider the system shown in Fig. 2. Assume that the **voltage source provides balanced positive sequence voltage phasors**, with the phase *a* voltage phasor $V_s = 1\angle 0°$ per unit. Also assume that the impedances are equal to $Z = 0 + j1$ per unit. Observe that the wye-connected voltage source is grounded through an impedance Z.

Problem 3 (25pts)

For the system shown in Fig. 2, compute each of the following fault types. Assume that each of the faults is *bolted* (that is, zero fault impedance to ground).

1. Compute the fault current $I_{a'f}$ for a bolted fault from a' to ground (single line to ground fault on phase a). [10pts]
2. Compute the fault current flowing from b' to ground ($I_{b'f}$) for a bolted fault from b' and c' to ground (double line to ground fault on phases b and c). [10pts]
3. Compute the current flowing from phase a to ground during a three-line to ground bolted fault at points a', b', and c'. [5pts]

Summary of the overall approach:

1. Construct the positive, negative, and zero sequence circuits
2. Connect the sequence circuits at the point of the fault accoridng to the fault type:
   (a) Single line to ground: Connect positive, negative, and zero sequence circuits in series.
   (b) Double line to ground: Connect positive, negative, and zero sequence circuits in parallel.
   (c) Line to line: Connect positive and negative sequence circuits in parallel, zero sequence circuit is disconnected.
3. Find the phase a symmetrical component values. ("Solve the circuit") for I_{af}^{+}, I_{af}^{-}, I_{af}^{0}, and V_a^+, V_a^-, and V_a^0.
4. Convert back to phase coordinates with the transformation

$$\begin{bmatrix} I_{af} \\ I_{bf} \\ I_{cf} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} I_{af}^{0} \\ I_{af}^{+} \\ I_{af}^{-} \end{bmatrix}.$$

This is analagous to a Fourier transform. In the first step, we have done the up-front work of creating a much simpler circuit. After solving this simpler circuit in this coordinate system, we have to "pay the price" of converting back by solving a linear system.

SOLUTION: We first begin by forming the sequence networks, shown in Fig. 3.

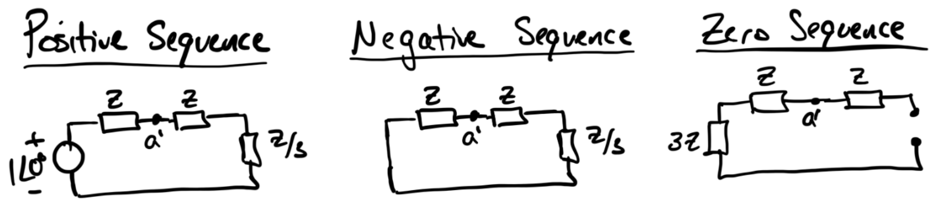

Figure 3: The sequence networks for problem 3

1. *Part 1—Single-line to ground fault at point a':* We connect the sequence networks in series at point a', and simplify, as shown in Fig. 4.

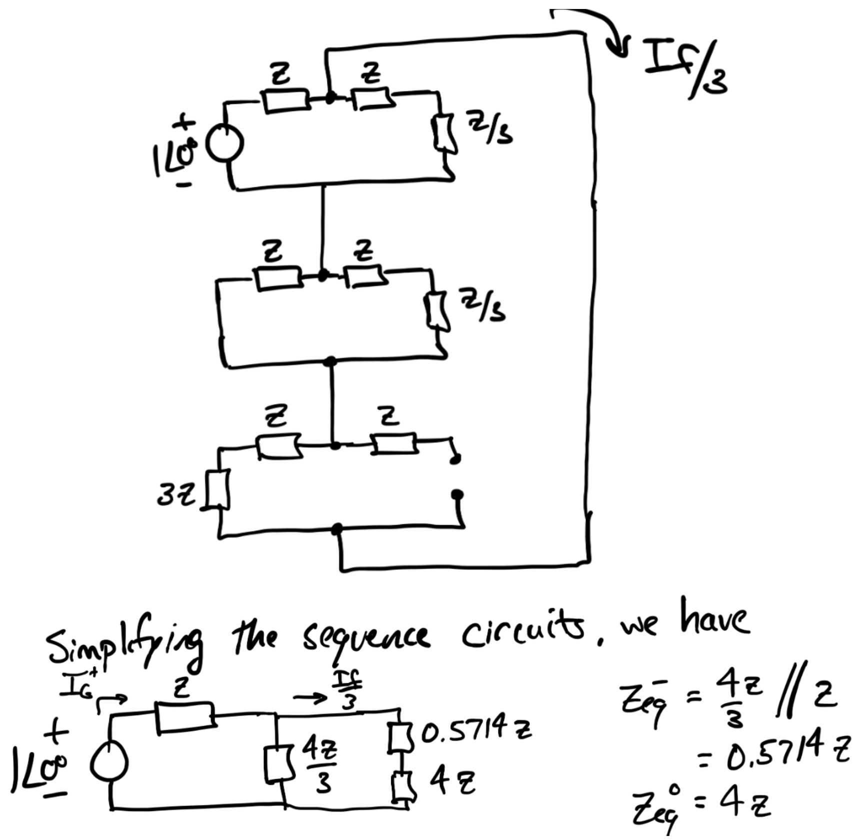

Figure 4: Sequence networks in series at point a' and the simplified circuit

Next, using the simplified circuit in Fig. 4, we obtain the generator current as:

$$I_G^+ = \frac{1\angle 0^\circ}{z + \left(\frac{4}{3}z /\!/ \,(4.5714)\,z\right)} = \frac{1\angle 0^\circ}{z\,(2.0323)} = -j0.4921;$$

thus, we can now solve for $I_f/3$ using a current divider:

$$\frac{I_f}{3} = I_G^+ \cdot \left(\frac{\frac{4}{3}z}{\frac{4}{3}z + 4.5714z} \right) = -j0.1111.$$

To wrap up,

$$\boxed{I_f = -j0.3333 = 0.3333\angle -90^\circ \quad \text{p.u.}}$$

2. *Part 2—Double-line to ground fault:* We connect the sequence networks **in parallel** at point a'; then, simplify the circuit to yield the result in Fig. 5.

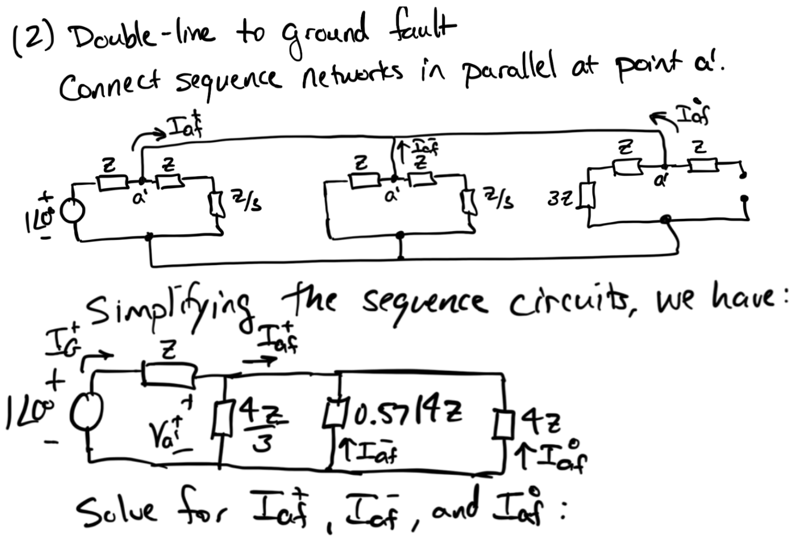

Figure 5: Sequence representation of the double line to ground fault and the simplified form

Next, we solve for I_{af}^+, I_{af}^-, and I_{af}^0. To do this, we first need the positive sequence generator current:

$$I_G^+ = \frac{1\angle 0^\circ}{z + \left(\frac{4}{3}z // 0.5714z // 4z\right)} = \frac{1\angle 0^\circ}{j1.3636} = -j0.733.$$

Thus,

$$V_{a'}^+ = 1\angle 0^\circ - I_G^+ \cdot z = 0.2667.$$

Consequently, the positive, negative, and zero-sequence fault currents are given as

$$I_{a'f}^+ = I_G^+ - \frac{V_{a'}^+}{4z/3} = -j0.7333 - \frac{0.2667}{j4/3} = -j0.5333$$

$$I_{a'f}^- = \frac{-V_{a'}^+}{j0.5714} = \frac{-0.2667}{j0.5714} = j0.4667$$

$$I_{a'f}^0 = \frac{-V_{a'}^+}{4z} = \frac{-0.2667}{j4} = j0.0667.$$

To wrap up, we need to perform the matrix-vector multiplication

$$\begin{bmatrix} I_{a'f} \\ I_{b'f} \\ I_{c'f} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} I^0_{a'f} \\ I^+_{a'f} \\ I^-_{a'f} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} j0.0667 \\ -j0.5333 \\ j0.4667 \end{bmatrix},$$

which, after plugging into a calculator, yields:

$$\boxed{\begin{bmatrix} I_{a'f} \\ I_{b'f} \\ I_{c'f} \end{bmatrix} = \begin{bmatrix} 0.0 \\ -0.8660 + j0.1000 \\ 0.8660 + j0.1000 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8718\angle 173.4^\circ \\ 0.8718\angle 6.6^\circ \end{bmatrix} \quad \text{p.u.}}$$

3. *Three-line-to-ground fault:* In this fault situation, the system is balanced and we can therefore write the single-phase equivalent circuit shown in Fig. 6.

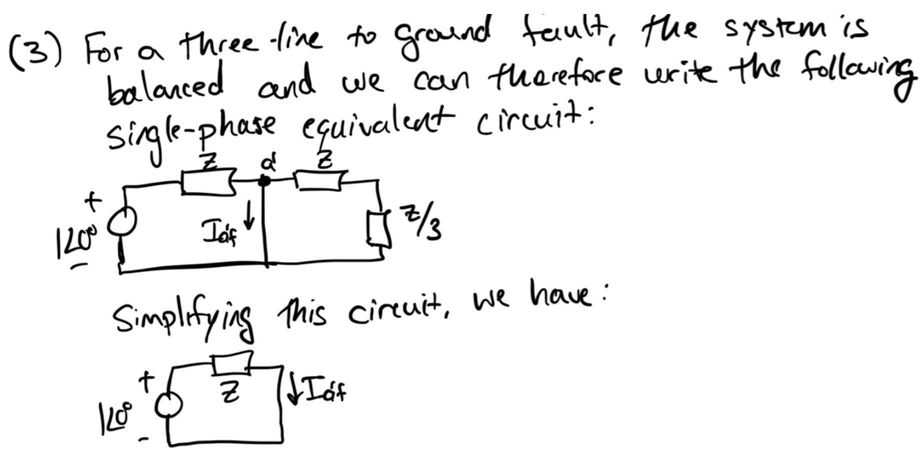

Figure 6: Three-line-to-ground fault single-phase equivalent circuit

Consequently, from Ohm's law, we have

$$I_{a'f} = \frac{1\angle 0^\circ}{z} = \frac{1\angle 0^\circ}{j1},$$

Thus,

$$\boxed{I_{a'f} = -j = 1\angle -90^\circ \quad \text{p.u.}}$$

4 Forming positive- negative-, and zero-sequence circuits from a one-line diagram

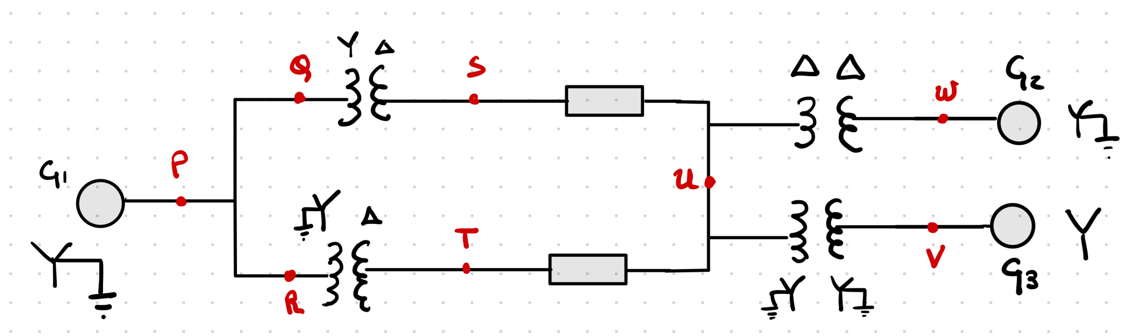

Figure 7: One-line diagram for the Problem 4 system.

Problem 4 (25pts)

Consider the one-line diagram shown in Fig. 7. The points P, Q, R, S, T, U, V, W label various points in this system. The positive-, negative-, and zero-sequence reactances for the generators are X_g^+, X_g^-, and X_g^0. Likewise, the transformers have positive, negative, and zero-sequence reances of X_T^+, X_T^-, and X_T^0, and the transmission lines (i.e., the lines connecting the points $S \to U$ and $T \to U$) have positive-, negative-, and zero-sequence reactances of X_l^+, X_l^-, and X_l^0.

Draw the positive-sequence, negative-sequence, and zero-sequence networks for the system in the one-line diagram. Label where the points P, Q, R, S, T, U, V, W, shown in Fig. 7, are equivalently located within your sequence networks.

Hints:

- *Note that the wye connection in the the transformer between points Q and S is not grounded.*
- *The generators G_1 and G_2 have a grounded wye connection, and G_3 is an ungrounded wye.*
- *Be sure to include the appropriate phase shifts for the transformers as needed.*

Sequence models for impedances

Let's recall the sequence representation of circuit components.

1. **Delta-connected impedances:** For a simple (delta-connected) impedance structure, we can write
$$\begin{bmatrix} I_a^0 \\ I_a^+ \\ I_a^- \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}.$$
Thus, we can write the zero-sequence current injection as
$$I_a^0 = \frac{1}{3}\left(I_a + I_b + I_c\right) = 0.$$
Physically, this means that *delta-connected impedances* are open circuits for the zero-sequence circuit. This also leads us to a nice intuition that **delta connections "block" zero-sequence currents**.

   In contrast, the positive and negative sequence impedances for delta-connected loads are just like what we saw at the beginning of the semester–the positive sequence transformation of a delta-connected impedance is $Z_\Delta/3$ with a delta-to-wye conversion; the same holds for the negative sequence.

2. **Wye-connected impedances:** We're familiar with these. The positive and negative sequence circuit maintains the same phase-coordinate impedance Z_Y. For the zero-sequence circuit, the impedance becomes $Z_y + 3Z_g$. Why do we have this? By KCL:
$$V_a = Z_y I_a^0 + Z_g\left(I_a^0 + I_b^0 + I_c^0\right) = Z_y I_a^0 + 3Z_g I_a^0 = I_a^0\left(Z_y + 3Z_g\right)$$

Sequence models for generators

1. Generators only produce positive sequence voltage in practice. It is possible for this generalization to not hold, but power engineers work very hard to make sure this doesn't happen.

2. In practice as a power system engineer, you will encounter specified values for the positive, negative, and zero-sequence reactances $Z^+ \approx Z^0 \approx jX_d^0$; zero-sequence impedance Z^0 usually smaller.

3. Specified value for the grounding impedance Z_g.

SOLUTION:

1. The positive sequence circuit is given as shown in Fig. 8.

2. The negative sequence circuit is given as shown in Fig. 9.

3. The zero-sequence circuit is given as shown in Fig. 10.

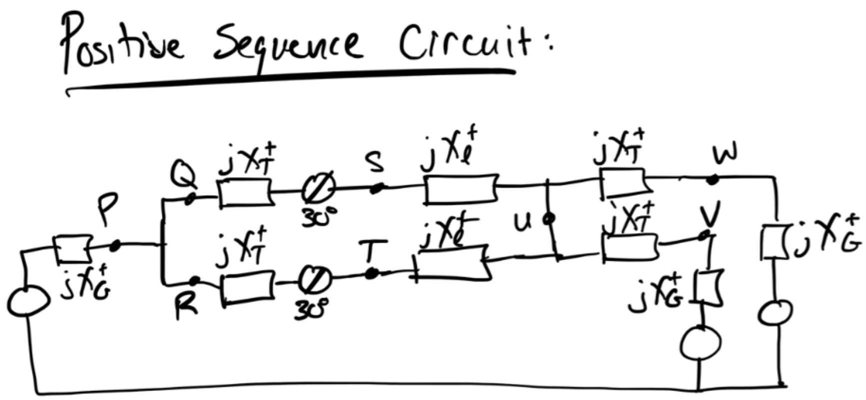

Figure 8: The positive sequence representation

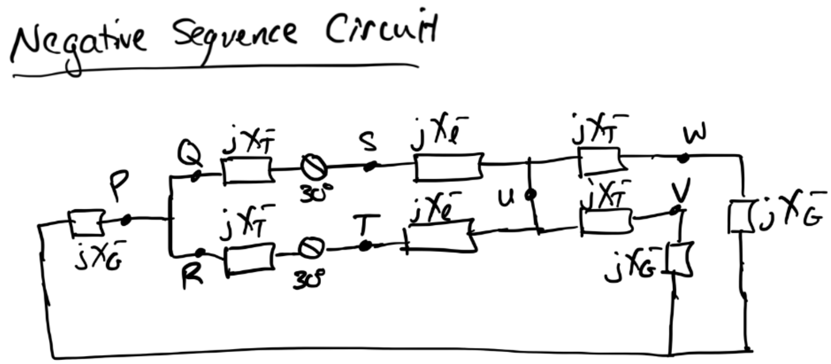

Figure 9: The negative sequence representation

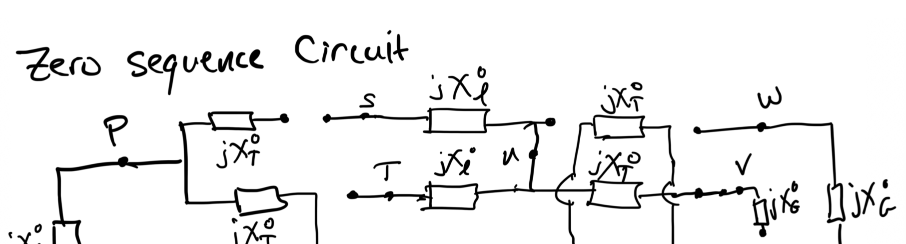

Figure 10: The zero sequence representation