Georgia Tech College of Engineering
School of Electrical
and Computer Engineering

ECE 2020

Number Systems: Rebooted

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October 8, 2024

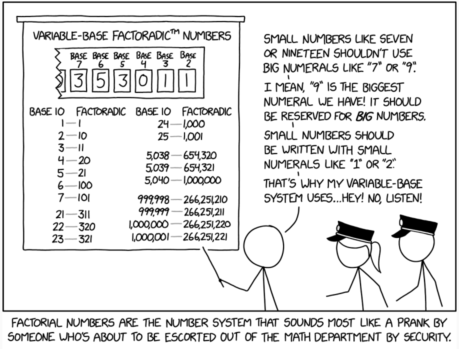
Logistics

- **HW3**: Due Wednesday, October 9th or Friday, October 11th.
- **Exam 1:**
  - Correction opportunity in office hours closes this week.
  - You should have received a message from me on Canvas if you are eligible.
  - If these times don't work for you, feel free to reach out.
- **Exam 2:** October 17th.
- **Midterm survey:** complete for +1 bonus point on your participation grade.

Agenda

Agenda: next 2 weeks

- Circuit timing
- Number systems
- Encoders/decoders
- Multiplexers
- Adders and subtractors



Source: xkcd

Numbers: How do they work?

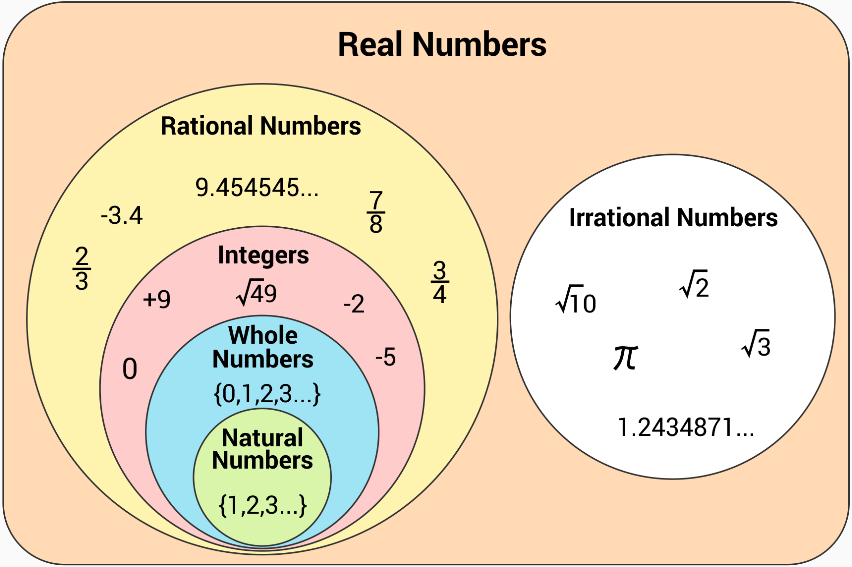


Figure 1: The subsets of the real numbers

Figure 2: Oldest known base-10 multiplication table, China, c. 305 BC

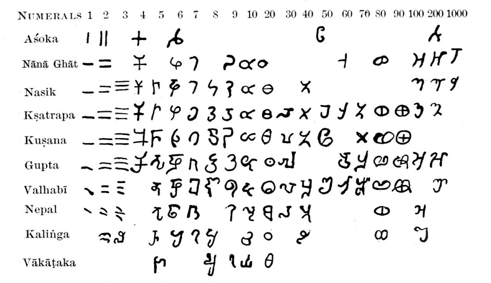


Figure 3: Evolution of Hindu-Arabic numerials, starting with Edicts of Ashoka, c. 250 BC

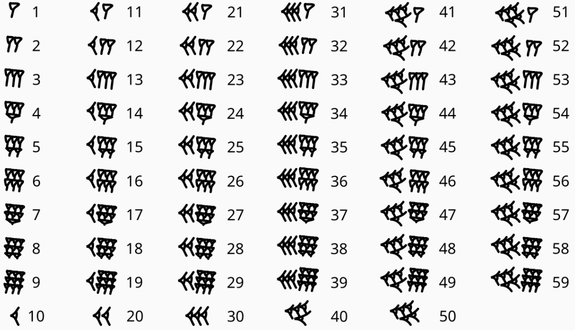


Figure 4: Babylonian cuneiform numerals, c. 2000 BC

The Babylonian cuneiform numerals, c. 2000 BC, were the first positional number system; shockingly, they were **base-60**, or *sexagesimal*. (???)

Bottom line

How we represent numbers is a *choice of human definition.*

Classroom discussion

There's an infinite number of real numbers, ("uncountably" infinite), and an infinite number of natural numbers, ("countably" infinite). Yet, we represent everything in terms of *just 10* of the natural numbers: $0, 1, \ldots, 9$.

Think about it for a second:

- How can we store numbers in computers?
- What kind of numbers would fit well into digital logic design?

Positional number systems

Question: How exactly do we represent a number?

Answer: We have to agree on the total number of **unique**, or **base** numbers to build our numbers from; the number of such unique numbers is called the *radix*.

$$\text{usual digits} = \underbrace{\{0, 1, 2, \ldots, 9\}}_{\#\text{digits}=\text{b},}$$

The total number of unique digits in a number system, *b*, is called the radix.

How? Positional numbers work by exponentiating the radix, multiplying the value of its place, and summing all of these together.

Example:

$$(241)_{10} = \left(2 \times 10^2\right) + \left(4 \times 10^1\right) + \left(1 \times 10^0\right)$$

We can make this more general!

Definition: Base-b number system

A *base-b number system* with *radix* $b > 1$ represents any number $x \in \mathbb{R}$ as a *string of digits* a_i in n "places" $i = 0, 1, \ldots, n-1$, where each a_i is one of b possible digits in a digit set $\mathcal{D}$:

$$a_i \in \mathcal{D} = \{d_1, d_2, \ldots, d_b\}.$$

Any real number x can be represented in a base-b system as the following sum:

$$x = (a_{n-1}a_{n-2} \ldots a_1 a_0)_b = \sum_{i=0}^{n-1} a_i \times b^i. \tag{1}$$

Base-10 number system

Base-10 numbers are the numbers we all know and love.

Examples:

- $\underbrace{3}_{=a_0} = 3 \times 10^0$
- $\underbrace{53}_{a_1 a_0} = 5 \times 10^1 + 3 \times 10^0$
- $\underbrace{125}_{=a_2 a_1 a_0} = \sum_{i=0}^{2} a_i \times 10^i = a_2 \times 10^2 + a_1 \times 10^1 + a_0 \times 10^0$

Base-2 (Binary) number system

Base-2 (a.k.a. binary) numbers are the numbers we are all (starting) to know and love.

Examples:

- $\underbrace{10}_{a_1 a_0} = 1 \times 2^1 + 0 \times 2^0 = (2)_{10}$
- $\underbrace{110}_{a_2 a_1 a_0} = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = (6)_{10}$
- $\underbrace{11010}_{a_4 a_3 a_2 a_1 a_0} = \sum_{i=0}^{3} a_i \times 2^i = 2^4 + 2^3 + 0 + 2^1 + 0 = (26)_{10}$

Base-16 (Hexadecimal) number system

In **base-16** or *hexadecimal* numbers, the set of base digits are:

$$\mathcal{D} = \{0, 1, 2, \ldots, 9, A, B, C, D, E, F\},$$

where:

$$(A)_{10} = 10, \quad (B)_{10} = 11, \quad (C)_{10} = 12,$$
$$(D)_{10} = 13, \quad (E)_{10} = 14, \quad (F)_{10} = 15.$$

Why care about non-base-10?

- base-2 (binary): Digital logic, all of computing instruction are converted to this.
- base-8: 3-bit information (useful in analysis, prototyping)
- base-16: 4-bit information (**tons** of computer stuff)
  - 32-bit IP addresses are 8 digits
  - 32-bit CPU instructions are 8 digits
- base-60: Deciphering ancient Babylonian Cuneiform tablets (**essential**)

Converting between number systems

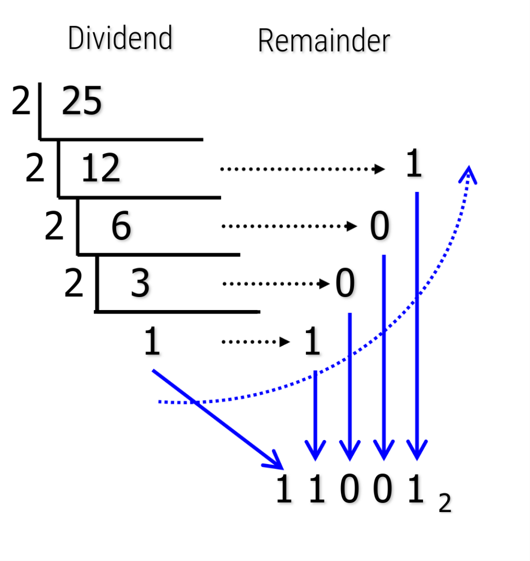


Figure 5: General procedure for converting $(25)_{10}$ to binary.

Converting between number systems i

To **convert** a base-*b* number to binary (base-2), you can follow these general concepts:

- Convert the number to base-10 using the appropriate number system.
- Divide the decimal number by 2.
- Note the remainder.
- Repeat the previous 2 steps for the quantient till the quotient is zero.
- Write the remainders in reverse order.

Example: converting a hexadecimal number to binary

How to convert this hexadecimal number to binary?

$$(4A)_{16} = (?)_2.$$

Solution: For the case of hex→binary, you can **individually convert** each digit into a **4-bit** binary digit.

$$(2A)_{16} = (42)_{10} = \underbrace{0010}_{=(4)_{10}}\ \underbrace{1010}_{(10)_{10}}$$

Examples i

Example

Convert:

$$(BEAD)_{16} = (?)_2$$

Answer:

$(BEAD)_{16} = (1011\ 1110\ 1010\ 1101)_2$

Examples ii

Example

Convert:

$$(10.1011001011)_2 = (?)_{16}$$

Answer:

$$(10.1011001011)_2 = (2.B2C)_{16}$$

Note the trailing zeros

Fractional number representations

Fixed-point fractional representation i

Consider the number 5.75 in base-10:

$$\underbrace{5}_{\text{whole part}}\underbrace{.}_{\text{decimal point}}\underbrace{75}_{\text{fractional part}}$$

Equivalently, in binary, $5.75 = 101.11$:

$$\underbrace{101}_{\text{whole part}}\underbrace{.}_{\text{binary point}}\underbrace{11}_{\text{fractional part}}$$

Fixed-point fractional representation ii

The reason for this is because:

$$(5)_{10} = (101)_2,$$

and

$$\begin{aligned}(0.75)_{10} &= (0.5)_{10} + (0.25)_{10} \\ &= 1 \times 2^{-1} + 1 \times 2^{-2}\end{aligned}$$

We can be put this in a general form:

Fractional number systems

A base b fractional number D with n whole number digits $D_{n-1}, D_{n-2}, \dots, D_0$ and r fractional digits $D_{-1}, D_{-2}, \dots, D_{-r}$, can be written as

$$D = (\underbrace{D_{n-1}D_{n-2} \dots D_1 D_0}_{n \text{ whole digits}} \, . \, \underbrace{D_{-1}D_{-2} \dots D_{-r}}_{r \text{ fractional digits}})_b,$$

and can be equivalently represented as

$$D = \sum_{i=-r}^{n-1} D_i b^i$$

Examples i

Example

Convert:

$$(100.101)_2 = (?)_{10}$$

Answer:

$$(100.101)_2 = (4.625)_{10}$$

Examples ii

Example

Convert:

$$(3A6.C)_{16} = (?)_2$$

Answer:

$$(3A6.C)_{16} = (\mathbf{00}11\,1010\,0110\,.\,11\mathbf{00})_2$$

Note the **trailing and leading** zeros above. Equivalently:

$$(3A6.C)_{16} = (11\,1010\,0110\,.\,11)_2$$

IMPORTANT: Trailing and Leading Zeros Rule

The rule for where to add 0's is *incredibly important* for correct conversions. For example, consider the conversion: $(22)_{10} = (10110)_2$. If we incorrectly add trailing zeros to convert this to hex, we would get

$$(10110000)_2 = (B0)_{16} = (176)_{10} \neq (22)_{10} \qquad \text{(false!)}$$

The correct way to convert this to hex is adding **leading zeros** to get

$$(00010110)_2 = (16)_{16} = (22)_{10} \qquad \text{(true!)}$$

IMPORTANT: Trailing and Leading Zeros Rule

Another example: $(2.5)_{10} = (10.1)_2$. If we want to convert this to hexadecimal, the correct way adding **leading *and* trailing zeros** around the decimal point:

$$(2.5)_{10} = (0010.1000)_2 = (1 \times 2^1) + (1 \times 2^{-1}) = (2.5)_{10} = (2.8)_{16}.$$

If we add trailing 0's to the right both before and after the decimal, we'll incorrectly get

$$(1000.1000)_2 = (8.8)_{16} = (8.5)_{10} \neq (2.5)_{10} \qquad \text{(false!)}$$

Furthermore, if we add leading zeros on both sides, we'll get

$$(0010.0001)_2 = (2.1)_{16} = (2.0625)_{10} \neq (2.5)_{10} \qquad \text{(false!)},$$

Binary addition

Binary addition i

Recall the basic rules:

- $0 + 0 = 0$
- $0 + 1 = 1 + 0 = 1$
- $1 + 1 = 10$

Binary addition ii

Example

Convert the following base-10 numbers into binary and perform the addition:

$$
\begin{aligned}
&(190)_{10} \\
+&(141)_{10} \\
=&(?)_2
\end{aligned}
$$

Answer:

$$(331)_{10}$$

Conversion:

$$
\begin{aligned}
&\;\;1\,0\,1\,1\,1\,1\,1\,0 \\
+&1\,0\,0\,0\,1\,1\,0\,1 \\
=&
\end{aligned}
$$

Binary addition iii

Example

Convert the following base-10 numbers into binary and perform the addition:

$$
\begin{aligned}
&(173)_{10} \\
+&(44)_{10} \\
=&(?)_2
\end{aligned}
$$

Answer:

$$(217)_{10} = (11011001)_2$$

How to do subtraction? Need negative number systems.

Negative number representations

Negative numbers i

Negative binary numbers are a system for representing negative numbers as binaries.

Signed magnitude binary representation:

- The **most significant bit** denotes the sign:
  - $0 \rightarrow$ positive number
  - $1 \rightarrow$ negative number
- **Example:**

$$(-25)_{10} = (1\,(25_{10})_2)_2 = (111001)_{2-\text{sgn}-\text{smag}}$$

Signed magnitude ii

More examples:

- $(+0)_{10} = (000)_{2-\text{sgn}-\text{mag}}$
- $(+1)_{10} = (001)_{2-\text{sgn}-\text{mag}}$
- $(+3)_{10} = (011)_{2-\text{sgn}-\text{mag}}$
- $(-3)_{10} = (110)_{2-\text{sgn}-\text{mag}}$

Signed magnitude iii

Why we don't use sign magnitude often:

- Representing zeros is tricky
- Complicated to use in circuits
- Other systems are easier to calculated
- We won't focus on this in this course, but you should be able to recognize them.

One's Complement Negative Number System

The **most significant bit** indicates the sign as in signed magnitude representations, **and**, if a number is negative (i.e., MSB $= 1$), then **all other bits are complemented.**

One's complement ii

Example

Convert into 1's complement binary representations:

$$(+25)_{10} = (?)_{1comp}$$
$$(-25)_{10} = (?)_{1comp}$$

Answer:

$$(+25)_{10} = (011001)_{1comp}$$
$$(-25)_{10} = (100110)_{1comp}$$

More examples of 1's complement:

- $(+3)_{10} = (011)_{1comp}$
- $(-3)_{10} = (100)_{1comp}$
- $(+2)_{10} = (010)_{1comp}$

Problem! Multiple representations for zero:

- $(+0)_{10} = (000)_{1comp}$
- $(-0)_{10} = (111)_{1comp}$

Problem! Arithmetic doesn't always work:

$(0)_{10} + (1)_{10} \stackrel{?}{=} \left((111)_{1comp} + (001)_{1comp}\right)_{10} = (0)_{10} \ldots$ ***False!***

Two's Complement

The *Two's Complement* representation of binary numbers is formed by taking the One's complement and adding 1, *ignoring any overflow*.

Primary method we will use in this course!

Two's complement ii

Why 2's complement?

- Arithmetic always works
- Unique representation for zero
- Support an extra negative value over 1's complement.

Procedure for reading a 2's complement number

- **Method 1:**
  (1) Complement using 2's complement
  (2) Find the value & place a negative sign
- **Method 2:**
  (1) Convert the value of all bits *except* the MSB as if it were a positive number
  (2) Let i_{MSB} be the place of the MSB. Subtract $2^{(\text{MSB})_{10}}$ from the converted number.

Example

Convert the following 2's complement number to base-10:

$$(101)_{2comp} = (?)_{10}$$

Answer: $(-3)_{10}$

Example

Convert the following 2's complement number to base-10:

$$(1010.1100)_{2comp} = (?)_{10}$$

Answer: $(-5.25)_{10}$

Arithmetic with 2’s complement

Addition, subtraction, multiplication

- Addition: same as before
- Subtraction: $A_{\text{2comp}} - B_{\text{2cmp}}$
  1.) Complement B using 2's complement
  2.) Add A and the complemented B together:

$$A - B = A + (-b)$$

- Multiplication: Put it in terms of repeated addition.

$$A \times B = \underbrace{A + A + \cdots + A}_{B \text{ times}}$$

(Other methods are available, but likely outside the scope of this class.)

Ranges of number systems

Consider an n-bit binary number X. We can represent the following rangles of base 10 numbers with each number system:

- Unsigned:
$$0 \leq (X)_{10} \leq 2^n - 1$$
- 1's complement:
$$-2^{n-1} - 1 \leq (X)_{10} \leq 2^{n-1} - 1$$
- 2's complement:
$$-2^{n-1} \leq (X)_{10} \leq 2^{n-1} - 1$$

Overflow

Definition: Overflow

If the result of an arithmetic operation exceeds the **range** of a number system, we say that **overflow** has occured.

WARNING

Overflow can **break arithmetic. Example:** $(5)_{10} + (6)_{10} = (?)_{2cmp}$

$$
\begin{aligned}
&(0101)_{2cmp} \\
+&(0110)_{2cmp} \\
=&(1011)_{2cmp} = (-5)_{10} \ldots \text{false! error from overflow.}
\end{aligned}
$$

Sign extensions

Sign extension: How to fix overflow

Sign extension is a procedure to modify a binary number without changing the value, to remove the risk of overflow.

How to fix overflow

Example: $(5)_{10} + (6)_{10} = (?)_{2cmp}$

$$
\begin{aligned}
&(00101)_{2cmp} \\
+&(00110)_{2cmp} \\
=&(01011)_{2cmp} = (+11)_{10}
\end{aligned}
$$

... correct! fixed by sign extension.

Overflow example

Overflow example

Perform this addition, note the overflow, and fix it with sign extension

$$
\begin{aligned}
&(-3)_{10} \\
+&(-6)_{10} \\
=&(?)_2
\end{aligned}
$$

Answer:

$$(110111)_{2cmp}$$