# Problem Set #5 (Bonus)

Due: Any time

2 problems, 10 points possible

## A note on the timeline/grading of this homework

1. Due to HW4 being extended a few times, this homework is being released later than planned.
2. This is a **bonus assignment; it is not required.** There is no penalty for not completing this.
3. If you complete this assignment, it will count as **extra credit** toward your homework grade.
4. Please **do not** stress yourself out working on this over Thanksgiving Break.

# Helpful information:

## Directed graphs

A state machine is a special case of a more general mathematical object called a **directed graph.**

A directed graph is described by a set of *n nodes* (in our case, these are states),

$$\mathcal{N} = \{0, 1, \ldots, n-1\},$$

and a set of *m edges*, which in our case, are the lines that connect the states:

$$\mathcal{E} \subseteq \{(u, v) \in \mathcal{N} \times \mathcal{N} \, : \, u \neq v\}.$$

One way to describe the connectivity of a graph is through a square *adjacency matrix* $\boldsymbol{A} \in \{0, 1\}^{n \times n}$, which has entries of the form

$$A_{ij} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are connected} \\ 0 & \text{otherwise.} \end{cases}$$

A state machine is a special form of a *weighted graph*, which adds a **weight function** $w : \mathcal{E} \to \mathbb{R}$, which assigns a number $w_{ij} = w(i, j)$ to each edge $(i, j) \in \mathcal{E}$. This creates a new matrix called a **weighted adjacency matrix** $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, which has entries that take the form

$$W_{ij} = \begin{cases} w(i, j) & \text{if node } i \text{ and node } j \text{ are connected} \\ 0 & \text{otherwise.} \end{cases}$$

## Markov chains are Moore state machines with probabilities

A *Markov chain* is a directed graph where the weighted adjacency matrix $\boldsymbol{W}$ has a special structure. The matrix $\boldsymbol{W}$ is a *stochastic matrix*; that is, it describes the **probability of moving from state $i$ to state $j$ in a single time step:**

$$W_{ij} = \text{Pr}\left(\text{moving from state } i \text{ to state } j\right) \qquad i, j = 1, \ldots, n.$$

We can think about a Markov chain as a **Moore state machine where the inputs are probabilities.** This also means that our weight matrix will have a special structure—each row of the matrix is a probability distirbution, so every row will have to sum up to one:

$$\sum_{j=1}^{n} W_{ij} = 1.$$

# Problem 1: Modeling city population distributions with state machines

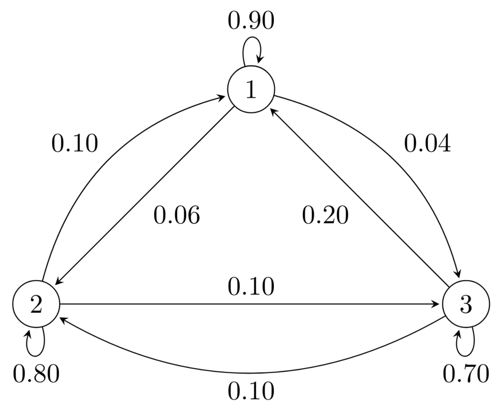


Figure 1: Three cities and the probability of people moving between them

**Problem 1** (8 points)

Consider a country with a total population $P$ that lives spread out among 3 cities: City 1, City 2, and City 3. Suppose that their populations are distributed as follows: 10% lives in City 1, 20% lives in City 2, and 70% lives in City 3. Suppose that the transitions are distributed according to the *Markov chain* shown in Fig. 1. This is a probabilistic version of a Moore state machine.

1. Derive a *transition matrix* (weighted adjacency matrix) for the graph shown in Fig. 1.

2. Define a *state vector* $\boldsymbol{x}_t \in [0,1]^3$ that describes the population distribution of each city at time step $t$, and let $\boldsymbol{x}_0$ be the initial state vector of the system at time $t = 0$, as described in the problem statement. Compute the state at the next timestep, $t = 1$ as

$$\boldsymbol{x}_1 = \boldsymbol{W}\boldsymbol{x}_0.$$

3. Compute more *state evolutions* by performing the matrix multiplications

$$\boldsymbol{x}_t = \boldsymbol{W}^t\boldsymbol{x}_0$$

multiple times for $t = 2$, $t = 3$. Where do the city populations settle as $t$ increases?

4. Use a computer or calculator to find the smallest value of $t$ such that $\boldsymbol{x}_t$ agrees with the equilibrium population distribution:

$$\boldsymbol{x}_\star = \lim_{t\to\infty} \boldsymbol{W}^t\boldsymbol{x}_0,$$

to 4 decimal places. Report the minimum number of time steps $t$ you need to achieve the equilibrium.

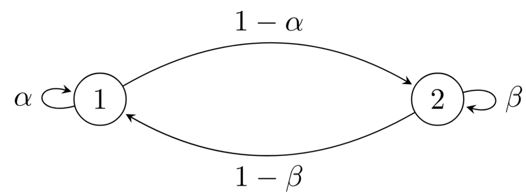


Figure 2: A generic Markov Chain

## Problem 2: Your first mathematical proof about Markov chains

**Definition 1** (Regular transition matrix)

A transition matrix (the weighted adjacency matrix) of a Markov chain is said to be *regular* if there exists a number $k$ such that every entry of $\boldsymbol{W}^k$ is greater than zero.

**Theorem 2** (When an equilibrium state vector exists)

If the transition matrix of a Markov Chain $\boldsymbol{W}$ is regular, than 1 is an eigenvalue of $\boldsymbol{W}$ and there exists an equilibrium state vector $\boldsymbol{x}_\star$ such that $\boldsymbol{x}_\star = \lim_{t\to\infty} \boldsymbol{W}^t \boldsymbol{x}_0$ for any initial distribution $\boldsymbol{x}_0$.

**Problem 2** (2 points, +2 bonus points possible)

Consider the generic Markov chain in Fig. 2

1. Write down the transition matrix for the Markov chain. Assuming that $0 < \alpha < 1$ and $0 < \beta < 1$, find the equilibrium state vector $\boldsymbol{x}_\star$ as $t \to \infty$.

2. **[Bonus (2pts)]** If we remove the assumption $\alpha$ and $\beta$, does an equilibrium state vector $\boldsymbol{x}_\star$ necessarily have to exist? If it does not have to, when can it exist? Justify your answer by providing a mathematical proof.