Linear power flow model

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May 17, 2024

Problem setup

Notation

- $\mathrm{j} := \sqrt{-1}$
- $\mathbf{I}_{n \times n}$ $n \times n$ identity matrix
- $\mathbf{1}_n$ n-dimensional vector of all ones
- $\boldsymbol{w} := \left[w_{ij}\right]_{(i,j) \in \mathcal{E}}$ network parameters; graph edge weights
- $\boldsymbol{A} \in \{0, \pm 1\}^{m \times n}$ edge-to-node incidence matrix
- $\mathrm{diag}(\cdot)$ diagonal matrix with the argument as entries
- $\{\cdot\} \circ \{\cdot\}$ elementwise multiplication of two vectors.

Admittance matrix

The *admittance matrix* $\boldsymbol{Y} := \boldsymbol{G} + \mathrm{j}\boldsymbol{B} \in \mathbb{C}^{n \times n}$ is a generalization of the *weighted graph Laplacian matrix*. It is a complex, symmetric, but *not necessarily Hermitian* matrix, and its entries take the form

$$Y_{ik} = \begin{cases} w_{ii} + \sum_{l \neq i} w_{il} & i = k \\ -w_{ik} & \text{otherwise,} \end{cases} \tag{1}$$

or equivalently,

$$G_{ik} + \mathrm{j}B_{ik} = \begin{cases} \left(g_{ii} + \sum_{l \neq i} g_{il}\right) + \mathrm{j}\left(b_{ii} + \sum_{l \neq i} b_{il}\right) & i = k \\ -g_{ik} - \mathrm{j}b_{ik} & \text{otherwise.} \end{cases} \tag{2}$$

Power flow equations

The *power flow equations* are a non-linear system of equations that describe Kirchhoff's current and voltage laws jointly. This system of equations is written as

$$\boldsymbol{s} := \boldsymbol{p} + \mathrm{j}\boldsymbol{q} = \operatorname{diag}(\boldsymbol{u})\, \underline{\boldsymbol{Y}\boldsymbol{u}}, \tag{3}$$

where $\underline{\{\cdot\}}$ is the complex conjugate, $\boldsymbol{u} \in \mathbb{C}^n$ are the *bus voltage phasors* and $\boldsymbol{Y} \in \mathbb{C}^{n\times n}$ is the admittance matrix.

Much ink has been spilled over solving these equations efficiently.

Constructing the power flow manifold

The power flow manifold

Define the *network state* in $\mathbb{R}^{4n}$ as

$$\boldsymbol{x} := \begin{bmatrix} \boldsymbol{v}^\mathsf{T} & \boldsymbol{\theta}^\mathsf{T} & \boldsymbol{p}^\mathsf{T} & \boldsymbol{q}^\mathsf{T} \end{bmatrix}^\mathsf{T}; \tag{4}$$

the power flow equations define a nonlinear operator $\mathcal{F} : \mathbb{R}^{4n} \to \mathbb{R}^{2n}$, where

$$\mathcal{F}(\boldsymbol{x}) := \begin{bmatrix} \mathrm{Re}\left\{\mathrm{diag}\left(\boldsymbol{u}\right) \underline{\boldsymbol{Y}\boldsymbol{u}} - \boldsymbol{s}\right\} \\ \mathrm{Im}\left\{\mathrm{diag}\left(\boldsymbol{u}\right) \underline{\boldsymbol{Y}\boldsymbol{u}} - \boldsymbol{s}\right\} \end{bmatrix} \tag{5}$$

with $\boldsymbol{u} := \boldsymbol{v} \circ \exp\{\mathrm{j}\boldsymbol{\theta}\}$ as the voltage phasors and $\boldsymbol{s} := \boldsymbol{p} + \mathrm{j}\boldsymbol{q}$ as the complex power injections. The *power flow manifold* is then

$$\mathcal{M} = \left\{\boldsymbol{x} \in \mathbb{R}^{4n} : \mathcal{F}(\boldsymbol{x}) = \boldsymbol{0}_{2n}\right\}. \tag{6}$$

Linearized power flow manifold

The *linear manifold tangent* to $\mathcal{M}$ at a nominal operating point $\boldsymbol{x}_\bullet$ is given by

$$\mathcal{M}_\bullet := \left\{ \boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\bullet)\,(\boldsymbol{x} - \boldsymbol{x}_\bullet) = \boldsymbol{0}_{2n} \right\}, \tag{7}$$

where

$$\boldsymbol{F}(\boldsymbol{x}_\bullet) := \frac{\partial \mathcal{F}}{\partial \boldsymbol{x}}(\boldsymbol{x}_\bullet) = \begin{bmatrix} \frac{\partial \mathcal{F}}{\partial \boldsymbol{v}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{\theta}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{p}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}}(\boldsymbol{x}_\bullet) \end{bmatrix} \tag{8a}$$

$$= \begin{bmatrix} \mathrm{Re}\left\{ \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) \right\} & \mathrm{Re}\left\{ \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) \right\} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathrm{Im}\left\{ \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) \right\} & \mathrm{Im}\left\{ \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) \right\} & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix} \tag{8b}$$

$$= \begin{bmatrix} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix}. \tag{8c}$$

Flat start linearization

Linearization of the power flow manifold [1, 2, 3]

Consider the *flat start condition* $\boldsymbol{u}_\star := \mathbf{1} + \mathrm{j}\mathbf{0}$, and suppose that $\boldsymbol{\omega} = \mathbf{0} + \mathrm{j}\mathbf{0}$. Then, the linear power flow manifold around $\boldsymbol{u}_\star$ is

$$\mathcal{M}_\star := \left\{ \boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\star)(\boldsymbol{x} - \boldsymbol{x}_\star) = \mathbf{0}_{2n} \right\}, \tag{9}$$

where

$$\boldsymbol{F}(\boldsymbol{x}_\star) = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ -\boldsymbol{B} & -\boldsymbol{G} & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix}, \tag{10}$$

or equivalently,

$$\boxed{\boldsymbol{p} = \boldsymbol{G}\boldsymbol{\epsilon} - \boldsymbol{B}\boldsymbol{\theta}, \quad \text{and} \quad \boldsymbol{q} = -\boldsymbol{B}\boldsymbol{\epsilon} - \boldsymbol{G}\boldsymbol{\theta},} \tag{11}$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - \mathbf{1}$.

Flat start linearization, part 1

Let $\boldsymbol{\omega} := \boldsymbol{\gamma} + \mathrm{j}\boldsymbol{\beta} \in \mathbb{C}^n$ denote the vector of self-admittances of each node. Then, following [5, 5.10]

$$
\begin{aligned}
\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\star) &= \mathrm{j}\,\mathrm{diag}\left(\boldsymbol{u}_\star\right)\left(\mathrm{diag}\left(\underline{\boldsymbol{Y}\boldsymbol{u}_\star}\right) - \underline{\boldsymbol{Y}}\,\mathrm{diag}\left(\underline{\boldsymbol{u}_\star}\right)\right) \\
&= \mathrm{j}\mathbf{I}_{n\times n}\left(\mathrm{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) - \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right) \\
&= \mathrm{j}\left(\mathrm{diag}\left(\underline{\boldsymbol{\omega}}\right) - \underline{\boldsymbol{Y}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\star) &= \mathrm{diag}\left(\boldsymbol{u}\right)\left(\mathrm{diag}\left(\underline{\boldsymbol{Y}\boldsymbol{u}}\right) + \underline{\boldsymbol{Y}}\,\mathrm{diag}\left(\underline{\boldsymbol{u}}\right)\right)\mathrm{diag}\left(\boldsymbol{v}\right)^{-1} \\
&= \mathbf{I}_{n\times n}\left(\mathrm{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) + \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right)\mathbf{I}_{n\times n}^{-1} \\
&= \mathrm{diag}\left(\underline{\boldsymbol{\omega}}\right) + \underline{\boldsymbol{Y}}
\end{aligned}
$$

Flat start linearization, part 2

We have that

$$
\begin{aligned}
\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\star) &:= \mathrm{Re}\left\{\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\star)\right\} \\
&= \mathrm{Re}\left\{\mathrm{j}\left(\mathrm{diag}\left(\underline{\boldsymbol{\omega}}\right) - \underline{\boldsymbol{Y}}\right)\right\} \\
&= \mathrm{Re}\left\{\mathrm{j}\left(\mathrm{diag}\left(\gamma - \mathrm{j}\beta\right) - \left(\boldsymbol{G} - \mathrm{j}\boldsymbol{B}\right)\right)\right\} \\
&= \mathrm{Re}\left\{\mathrm{j}\,\mathrm{diag}(\gamma) + \mathrm{diag}(\beta) - \mathrm{j}\boldsymbol{G} - \boldsymbol{B}\right\} \\
&= \mathrm{diag}\left(\beta\right) - \boldsymbol{B},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\star) &:= \mathrm{Re}\left\{\mathrm{diag}\left(\underline{\boldsymbol{\omega}}\right) + \underline{\boldsymbol{Y}}\right\} \\
&= \mathrm{Re}\left\{\mathrm{diag}\left(\gamma - \mathrm{j}\beta\right) + \boldsymbol{G} - \mathrm{j}\boldsymbol{B}\right\} \\
&= \mathrm{diag}\left(\gamma\right) + \boldsymbol{G}.
\end{aligned}
$$

Flat start linearization, part 3

Similarly,

$$
\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}\left(\boldsymbol{u}_{\star}\right) &= \operatorname{Im}\left\{\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}\left(\boldsymbol{u}_{\star}\right)\right\} \\
&= \operatorname{Im}\left\{\mathrm{j}\left(\operatorname{diag}\left(\underline{\boldsymbol{\omega}}\right) - \underline{\boldsymbol{Y}}\right)\right\} \\
&= \operatorname{Im}\left\{\mathrm{j}\operatorname{diag}(\gamma) + \operatorname{diag}(\beta) - \mathrm{j}\boldsymbol{G} - \boldsymbol{B}\right\} \\
&= \operatorname{diag}\left(\gamma\right) - \boldsymbol{G}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}\left(\boldsymbol{u}_{\star}\right) &= \operatorname{Im}\left\{\operatorname{diag}\left(\underline{\boldsymbol{\omega}}\right) + \underline{\boldsymbol{Y}}\right\} \\
&= \operatorname{Im}\left\{\operatorname{diag}\left(\gamma - \mathrm{j}\beta\right) + \boldsymbol{G} - \mathrm{j}\boldsymbol{B}\right\} \\
&= -\operatorname{diag}\left(\beta\right) - \boldsymbol{B}
\end{aligned}
$$

Applying the assumption that $\beta = \gamma = \mathbf{0}$ and plugging into (8) yields the desired result:

$$F(\mathbf{x}_\star) = \begin{bmatrix} G & -B & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ -B & -G & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix}. \tag{12}$$

Inverse model for tree networks

Remark: Model inversion

In the special case of ***radial (tree) networks***, i.e., the practically relevant setting of small-scale ***distribution networks*** (the main application setting of interest), a reduced form of the linear model (11) can be inverted in closed form.

Trees only

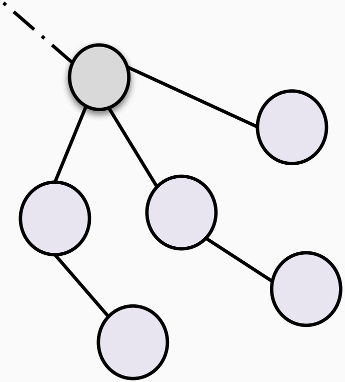

Figure 1: Hereafter, everything only works for tree networks

Reduced incidence matrix

With an abuse of notation, for a ***tree network*** with n lines and n non-reference nodes, $\mathcal{N} := \{0, 1, \dots, n\}$, let $\boldsymbol{A} \in \{0, \pm 1\}^{n \times n}$ be the reduced incidence matrix formed by removing the first column of the full incidence matrix, which we now denote as

$$\boldsymbol{A}_\star := \begin{bmatrix} \boldsymbol{a}_0 & \boldsymbol{A} \end{bmatrix} \in \{0, \pm 1\}^{n \times (n+1)} . \tag{13}$$

It is known that the reduced incidence matrix $\boldsymbol{A}$ is square and invertible [4].

Reduced Laplacian matrices

With another abuse of notation, keep the same formula for the admittance matrix $\boldsymbol{Y}$,

$$\boldsymbol{Y} := \boldsymbol{A}^\top \operatorname{diag}(\boldsymbol{w})\boldsymbol{A} = \boldsymbol{A}^\top \operatorname{diag}(\boldsymbol{g})\boldsymbol{A} + \mathrm{j}\boldsymbol{A}^\top \operatorname{diag}(\boldsymbol{b})\boldsymbol{A}, \tag{14}$$

and define the following reduced graph Laplacian matrices:

$$\boldsymbol{G} := \operatorname{Re}\{\boldsymbol{Y}\} = \boldsymbol{A}^\top \operatorname{diag}(\boldsymbol{g})\boldsymbol{A}, \quad \text{and} \quad \boldsymbol{B} := \operatorname{Im}\{\boldsymbol{Y}\} = \boldsymbol{A}^\top \operatorname{diag}(\boldsymbol{b})\boldsymbol{A}, \tag{15}$$

where $\boldsymbol{A}$ is now the *reduced incidence matrix*.

Now the inverse of $\boldsymbol{Y}$, the *impedance matrix*, can be computed as

$$\boldsymbol{Y}^{-1} = \boldsymbol{A}^{-1} \operatorname{diag}(\boldsymbol{w})^{-1}\boldsymbol{A}^{-\top} = \boldsymbol{R} + \mathrm{j}\boldsymbol{X}. \tag{16}$$

Intuitive explanation

Recall that for any complex number ζ, $\zeta^{-1} = \underline{\zeta} / |\zeta|^2$. Hence, the impedances can be expressed as inverse admittances (graph weights):

$$w_{ij}^{-1} := r_{ij} + \mathrm{j}x_{ij} := \underbrace{\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}}_{:= r_{ij}} + \mathrm{j} \underbrace{\frac{-b_{ij}}{g_{ij}^2 + b_{ij}^2}}_{:= x_{ij}} \quad \forall (i,j) \in \mathcal{E}, \tag{17}$$

thus, define the matrices

$$\boldsymbol{R} = \boldsymbol{A}^{-1} \operatorname{diag}(\boldsymbol{r}) \boldsymbol{A}^{-\mathsf{T}}, \quad \boldsymbol{X} := \boldsymbol{A}^{-1} \operatorname{diag}(\boldsymbol{x}) \boldsymbol{A}^{-\mathsf{T}}.$$

Inverse model for tree networks

Consider a tree network with n non-reference nodes and n edges. Then,

$$\begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix}, \tag{18}$$

where

$$\boldsymbol{R} = \boldsymbol{A}^{-1}\operatorname{diag}(\boldsymbol{r})\boldsymbol{A}^{-\top},$$
$$\boldsymbol{X} := \boldsymbol{A}^{-1}\operatorname{diag}(\boldsymbol{x})\boldsymbol{A}^{-\top}.$$

and thus

$$\boxed{\boldsymbol{\epsilon} = \boldsymbol{R}\boldsymbol{p} + \boldsymbol{X}\boldsymbol{q}, \quad \text{and} \quad \boldsymbol{\theta} = \boldsymbol{X}\boldsymbol{p} - \boldsymbol{R}\boldsymbol{q},} \tag{19}$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - 1$.

Inverting the model for trees

We want to invert the system of equations

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} G & -B \\ -B & -G \end{bmatrix} \begin{bmatrix} \epsilon \\ \theta \end{bmatrix}. \tag{20}$$

Note that as $G \succ 0$, both Schur complements of the above matrix exist, and hence

$$\begin{bmatrix} G & -B \\ -B & -G \end{bmatrix}^{-1} = \begin{bmatrix} \left(G + BG^{-1}B\right)^{-1} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & -\left(G + BG^{-1}B\right)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n\times n} & -BG^{-1} \\ BG^{-1} & \mathbf{I}_{n\times n} \end{bmatrix}$$

$$= \begin{bmatrix} \left(G + BG^{-1}B\right)^{-1} & -\left(G + BG^{-1}B\right)^{-1} BG^{-1} \\ -\left(G + BG^{-1}B\right)^{-1} BG^{-1} & -\left(G + BG^{-1}B\right)^{-1} \end{bmatrix}.$$

Inverting the model for trees

To begin, the first matrix we need to compute is

$$
\begin{aligned}
\left(\boldsymbol{G} + \boldsymbol{B}\boldsymbol{G}^{-1}\boldsymbol{B}\right)^{-1} &= \left(\boldsymbol{A}^{\mathsf{T}} \operatorname{diag}(\boldsymbol{g})\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}} \operatorname{diag}(\boldsymbol{b})\boldsymbol{A}\boldsymbol{A}^{-1} \operatorname{diag}(\boldsymbol{g})^{-1}\boldsymbol{A}^{-\mathsf{T}}\boldsymbol{A}^{\mathsf{T}} \operatorname{diag}(\boldsymbol{b})\boldsymbol{A}\right)^{-1} \\
&= \left(\boldsymbol{A}^{\mathsf{T}} \operatorname{diag}\left(\left[\frac{g_{ij}^2 + b_{ij}^2}{g_{ij}}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}\right)^{-1} \\
&= \boldsymbol{A}^{-1} \operatorname{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\mathsf{T}} \\
&:= \boldsymbol{A}^{-1} \operatorname{diag}\left(\boldsymbol{r}\right)\boldsymbol{A}^{-\mathsf{T}} \\
&:= \boldsymbol{R}.
\end{aligned}
$$

Inverting the model for trees

Finally, the second matrix we need to compute is

$$
\begin{aligned}
-\left(\boldsymbol{G}+\boldsymbol{B}\boldsymbol{G}^{-1}\boldsymbol{B}\right)^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} &= -\boldsymbol{A}^{-1}\operatorname{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2+b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top}\boldsymbol{A}^{\top}\operatorname{diag}(\boldsymbol{b})\boldsymbol{A}\boldsymbol{A}^{-1}\operatorname{diag}(\boldsymbol{g})^{-1}\boldsymbol{A}^{-\top} \\
&= \boldsymbol{A}^{-1}\operatorname{diag}\left(\left[\frac{-b_{ij}}{g_{ij}^2+b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top} \\
&:= \boldsymbol{A}^{-1}\operatorname{diag}\left(\boldsymbol{x}\right)\boldsymbol{A}^{-\top} \\
&:= \boldsymbol{X}
\end{aligned}
$$

The matrices of the inverted model

Then, the matrix inverse we have obtained is

$$\begin{bmatrix} G & -B \\ -B & -G \end{bmatrix}^{-1} = \begin{bmatrix} R & X \\ X & -R \end{bmatrix} \tag{21}$$

and thus

$$\begin{bmatrix} \epsilon \\ \theta \end{bmatrix} = \begin{bmatrix} R & X \\ X & -R \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \tag{22}$$

as desired.

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